Computational Morphology
G.T. Toussaint (editor)

UNIQUENESS OF ORTHOGONAL CONNECT-THE-DOTS

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It is proven that a collection of non-intersecting simple orthogonal polygons is uniquely determined by its vertex set, and reconstructible from this set in $O(n \log n)$ time. The theorem is also extended to three dimensions: a collection of orthogonal polyhedra is uniquely determined by its set of edges.

1. INTRODUCTION

Our perceptual system's ability to reconstruct the shape of objects from sparse partial data has led pattern recognition researchers to study a particularly pure condensation of the problem: given a set of points (dots) in the plane, connect them with polygonal paths into a "meaningful" whole. This is hopelessly vague as stated, but several natural constraints suggest themselves that lead to interesting problems and solutions [8] [3] [2] [7] [1]. One such constraint requires the dots to be connected to form simple closed polygons. This still leaves so much freedom, however, that it seems difficult to choose a "natural" candidate solution [5]. In search of a more constrained problem, I investigate here the problem of connecting dots into orthogonal polygons.[1] The conclusion is that there is at most one way of connecting a set of dots into a collection of orthogonal polygons. This theorem extends to three dimensions: there is at most one way to flesh out a wire-frame into a collection of orthogonal polyhedra.

More formally, let P be a collection of non-intersecting simple orthogonal polygons in the plane. Each polygon is composed of an alternating sequence of horizontal and vertical edges, any two of which meet in at most one point, which point is a vertex of the polygon. Each edge contains

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[1]First suggested by Derick Wood.

exactly two vertices; vertices are only permitted at corners. The polygons may be nested inside one another, but they may not intersect or share vertices. The vertices of the polygons are not necessarily in "general position": any number may fall on the same vertical or horizontal line. We will view P as composed of two sets: the vertices V and the edges E.

Now imagine erasing all the edges of P, leaving only the set of vertices V on the plane. The main result of this note is that E is uniquely determined by V. The curious implication is that the edges of orthogonal polygons are superfluous: there is only one way (if there is a way) to connect a set of dots into orthogonal polygons. Moreover, the proof of this theorem is constructive, leading to an $O(n \log n)$ algorithm for reconstructing P from V.

2. TWO DIMENSIONS

Each vertex is incident to exactly one horizontal edge and one vertical edge. A vertex that is leftmost on the horizontal line containing it must be the left endpoint of a horizontal edge, and a vertex that is uppermost on the vertical line containing it must be the upper endpoint of a vertical edge. These simple observations lead to the following algorithm.

Index the vertices on a particular horizontal line H from 1 to m; m must be even because each horizontal edge on H has two distinct endpoints. Let $[i,j]$ be the horizontal edge from vertex i to j on H. Then the edges $[2i-1,2i]$ for $1 \leq i \leq m/2$ must be in E, and these are the only edges in E that lie on H. $[1,2] \in E$ because 1 is leftmost; $[2,3] \notin E$ because 2 may only have one incident horizontal edge; $[3,4] \in E$ because 3 must have an incident horizontal edge; and so on. Therefore all the horizontal edges of P may be reconstructed by connecting alternate pairs of dots in each horizontal row as above. Similarly all vertical edges can be reconstructed by connecting alternate pairs of dots in each vertical column from top to bottom. Fig. 1 shows a particular point set and Fig. 2 shows its unique reconstruction.

This procedure can be accomplished easily in $O(n \log n)$ time by sorting the vertices horizontally and vertically.

Note that the algorithm also detects when a point set cannot be the vertex set of a collection of non-intersecting simple orthogonal polygons: whenever no move can be made at a step of the algorithm, or when edges determined by the algorithm cross. An example of the former case is shown if Fig. 3a, and of the latter case in Fig. 3b.

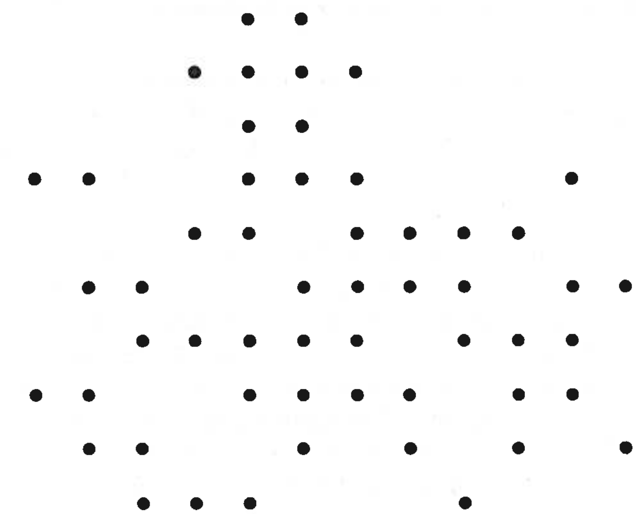

FIGURE 1
A set of dots in the plane.

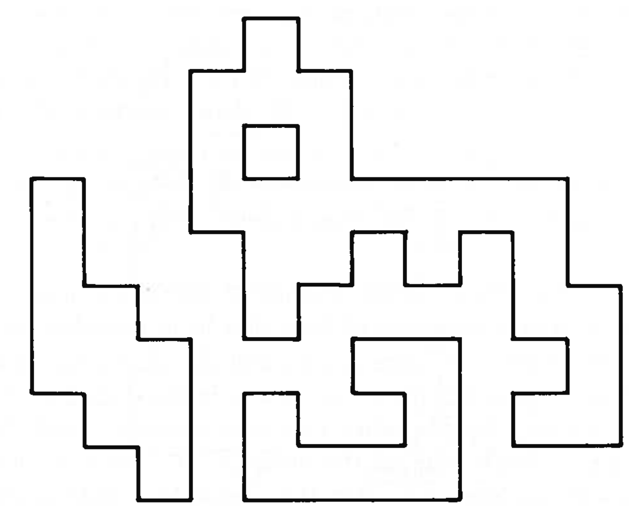

FIGURE 2
The orthogonal polygons reconstructed from Fig. 1.

The results of preceding discussion can be summarized in the following theorem:

THEOREM 1. A collection of non-intersecting orthogonal polygons is uniquely determined by its set of vertices V, and its edges can be reconstructed from V in $O(n \log n)$ time, where $|V| = n$.

3. THREE DIMENSIONS

This theorem cannot be extended directly to three (or higher) dimensions: two identical crosses arranged parallel, one above the other, can be connected by a central beam without addition of vertices. More specifically, reflect the three points { (2,1), (1,1), (1,2) } in the X-axis, the Y-axis, and the origin, resulting in 12 points defining a cross in the XY plane. Define V to be the 48 points composed of this pattern at $Z=0$, $Z=1$, $Z=2$, and $Z=3$. Then the central rectangular box defined by the square { (1,1), (-1,1), (-1,-1), (1,-1) } between $Z=1$ and $Z=2$ either may or may not be present; see Fig. 4. Thus V does not uniquely determine a collection of orthogonal polyhedra.

The crucial property used in the two-dimensional proof that no longer holds in three dimensions is that each vertex is incident to exactly one horizontal and one vertical edge. Call an edge parallel to the X-axis an X-edge, and similarly for Y- and Z-edges. Then in the $X=1$ plane, the vertex (1,1,1) in Fig. 4 has an incident Y-edge [(1,1,1), (1,2,1)], but may or may not have the incident Z-edge [(1,1,1), (1,1,2)], shown dotted in Fig. 4.

To resolve this ambiguity, it is necessary to know the edges as well as the vertices. Of course the edges determine the vertices by definition, so only the edges need be given for reconstruction. We now establish this claim.

Let P be a collection of simple orthogonal polyhedra in three dimensions. Each polyhedron is composed of faces that lie in planes parallel to two of three orthogonal coordinate axes X, Y, and Z. A plane parallel to the Y and Z axes is called a YZ-plane, and a face in the YZ-plane is called a YZ-face, and similarly for the other two combinations. Each face is an orthogonal polygon. Each edge on the boundary of a face is contained in exactly two orthogonal faces; note that there must be a right angle at each edge: edges are not permitted to lie at the junction of two coplanar faces. Distinct faces intersect only at edges and vertices, if at all. We view P as composed of three sets: the faces F, the edges E, and the vertices V.

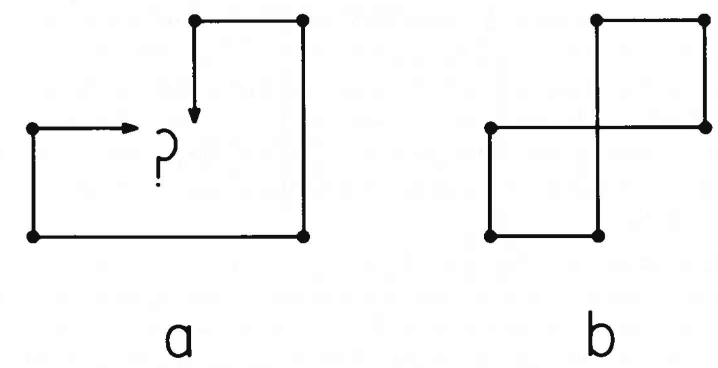

FIGURE 3
Two illegal reconstructions: (a) no matching vertex; (b) illegal self-intersection.

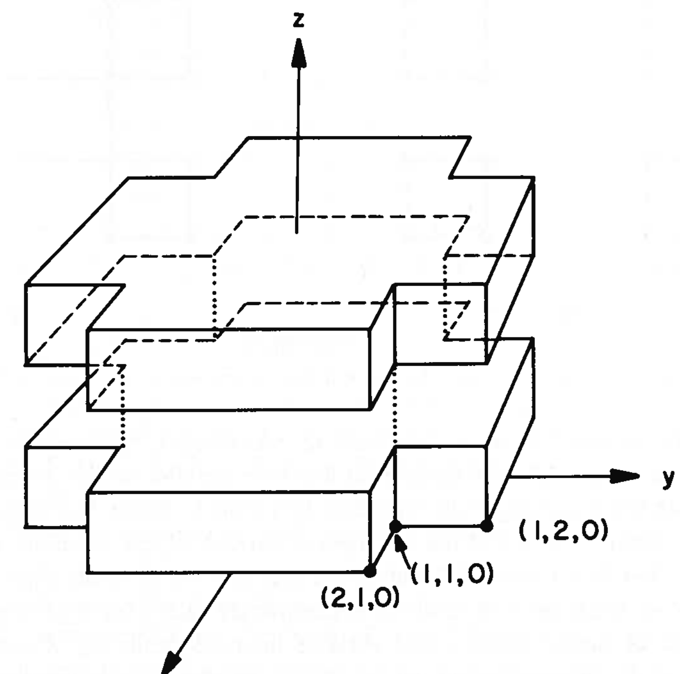

FIGURE 4
An example of 48 vertices in three dimensions that do not uniquely determine polyhedra: the central box between the two crosses (whose vertical edges are dotted) either may or may not be present.

Suppose we are given E, with each edge defined by a pair of endpoints. V is simply the union of all these endpoints. We reconstruct F from E in three parts. The first step computes the YZ-faces. Let $X=c$ be any YZ-plane π that wholly contains at least one edge $e \in E$. Since e is in the boundary of exactly two orthogonal faces, one of them, say $f \in F$, must lie wholly in π. Recall that f is an orthogonal polygon; so all its boundary edges lie wholly in π.

It is natural then to conclude that we have an instance of the two-dimensional problem just solved: a collection of orthogonal polygons in π. But the collection does not necessarily satisfy the assumption that each vertex is incident to exactly two edges. Returning again to Fig. 4, let π be the plane $X=1$. Then, if the central box is absent, the collection of edges in π is as shown in Fig. 5a; if the box is present, then as in Fig. 5b. In Fig. 5b, some vertices are incident to four edges.

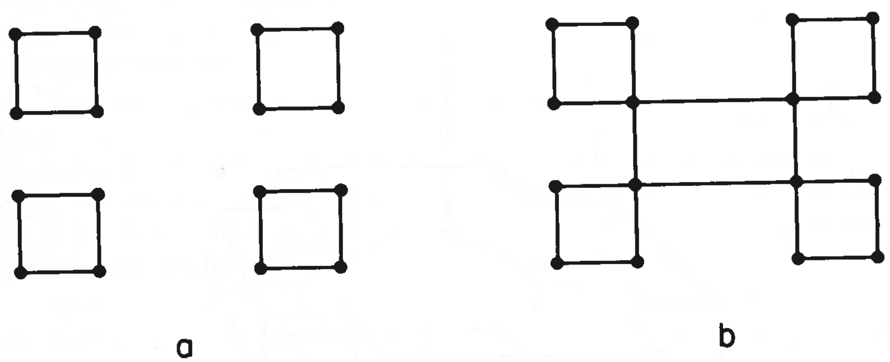

FIGURE 5

The edges that lie in $X=1$ in Fig. 4 if the central box is not (a) or is (b) present.

Nevertheless it is not difficult to reconstruct the faces in π from the edges in π. Let H be a horizontal line in π parallel to the Y-axis that intersects at least one edge but contains no vertices. Index the edges intersected by H from 1 to m. Now we have a variant of the problem solved previously. Let $[i,j]$ represent the closed segment on H from edge i to edge j. Again m must be even, and the segments $[2i-1,2i]$ for $1 \le i \le m/2$ must be subsets of faces. Sorting the vertices in π vertically by Z-coordinate and locating H between every adjacent pair in this sort, is clearly sufficient to identify the interior of each orthogonal face. See Fig. 6.

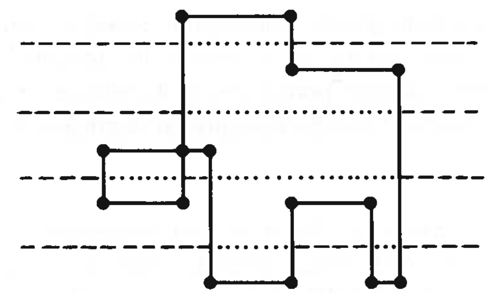

FIGURE 6

The interior of the faces may be detected by positioning a line between vertices; here dashed indicates exterior and dotted interior.

Reconstruction of all YZ-faces can then be achieved by setting π to each YZ-plane that contains an edge. The XZ- and XY-faces may be reconstructed similarly. Informally, the conclusion of this argument is that orthogonal polyhedra may be reconstructed from their "wire-frames." The following theorem is a more formal statement.

THEOREM 2. A collection of orthogonal polyhedra that intersect only at vertices (if at all) is uniquely determined by its set of edges E, and its faces may be reconstructed from E in $O(n \log n)$ time, where $|E|=n$.

Note that the polyhedra may intersect at vertices, as this does not destroy the crucial property that each edge is shared by exactly two orthogonal faces.

4. DISCUSSION

The generalization of this theorem to higher dimensions is that a collection of orthogonal polytopes in d-dimensions is uniquely reconstructible from its set of $(d-2)$-dimensional facets or "ridges." The proof of this claim is similar in spirit to those above and will not be detailed. Note that in higher dimensions, the number of these facets is no longer necessarily linear in the number of vertices. Perhaps of more interest is a question posed by Raimund Seidel: can the two-dimensional theorem be generalized to polygons with edges parallel to a given fixed set of directions? Exploration of this question may uncover a middle ground between the extreme freedom of the unconstrained problem, and the total lack of freedom of the orthogonally constrained problem.

In another direction, David Rappaport recently showed that if the assumption that each vertex is a corner is removed from the two-dimensional problem, thereby permitting two collinear edges to meet at a vertex, the reconstruction problem becomes NP-complete [6].

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ON THE SHAPE OF A SET OF POINTS

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Inspired by recent developments in computational morphology, this paper discusses their potential impact on research undertaken in the field of spatial analysis addressing the characterization and recognition of form in point sets. A brief discussion of point pattern recognition methods now common in spatial analysis is included, pointing to their limitations and questioning their success. The main focus of the paper, however, is the examination of recent methods of geometric decomposition that appear more useful for solving questions concerning form. New techniques based in computational morphology may very well revolutionize the characterization of point sets for spatial analysts. Some of these techniques are referenced and briefly discussed here, including their potential applications to point pattern recognition problems in spatial analysis.

1. Introduction

What at first appears to be a simple task, to address the shape of a set of points, becomes a complex and thought provoking endeavour when the points are locational identifiers of some real spatial phenomenon. Physical, biological and social scientists have always been faced with this challenge but progress seems to have been made in the past few decades. This paper identifies the use of some ideas and techniques of computational morphology that appear useful to scientists in a variety of disciplines.

Computational geometry is quickly expanding and influencing other fields of study. Computational morphology, according to Toussaint occurs when "a computational geometric structure is intended to extract the form of the input" [T3]. The input, usually representing a spatial distribution of some phenomenon, can be easily characterized by a set of elements, such as points, lines or areas. The recognition of spatial patterns within these

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