Polygons needing many flipturns

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Abstract

A *flipturn* on a polygon consists of reversing the order of segments inside a pocket of the polygon, without changing lengths or slopes. Any n-link polygon can be convexified by performing at most $(n-1)!$ flipturns. A very recent paper showed that in fact it is convex after at most $n(n-1)/2$ flipturns. We give here a lower bound by constructing a polygon such that if pockets are chosen in a bad way, at least $(n-2)^2/4$ flipturns are needed to convexify the polygon.

1 Background

We assume familiarity with polygons and 2D geometry. Assume that P is a 2D polygon that is not convex. A *pocket* of P is a set of contiguous links of P such that none of them is on the convex hull of P, but the two endpoints of the pocket are on the convex hull of P. Given a pocket, a *flip* of the pocket consists of reflecting the pocket along the line through the two endpoints of the pocket. A *flipturn* consists of reversing the order of links in the pocket. Thus, if $l_1, \dots, l_k$ are the links of the pocket, and $l_{k+1}, \dots, l_n$ are the remaining links of the polygon, then the polygon that results from doing a flipturn to this pocket consists of the links $l_k, l_{k-1}, \dots, l_1, l_{k+1}, l_{k+1}, \dots, l_n$, where none of the lengths or slopes of the links change. See Figure 1 for an example.

Flipturns were mentioned for the first time in a paper by Grünbaum [Grü95]. He reports that Joss and Shannon (unpublished) observed that any polygon is convexified after applying at most $(n-1)!$ flipturns; this follows because every flipturn creates a different permutation of the links, and no permutation can repeat. Also, Joss and Shannon conjectured that $n^2/4$ flipturns suffice to convexify a polygon. Very recently, Ahn et al. proved that $n(n-1)/2$ flipturns suffice to convexify any polygon, and in fact, if the polygon has only k different slopes, then $k(n-1)/2$ flipturns suffice [ABC$^+$00].

For related concept, and in particular the intricate history of flips (not flipturns) see [Tou99].

In this note, we provide a lower bound on the number of flipturns needed to convexify a polygon. More precisely, we construct a polygon such that if flipturns are chosen in a bad sequence, at least $(n-2)^2/4$ flipturns are needed to convexify the polygon. The constructed polygon however has links that are very long (exponentially long in n). We therefore give another construction, which needs at least $(n-2)(n-4)/8$ flipturns if the flipturns are chosen in a bad sequence, and where the link-lengths are polynomial in n.

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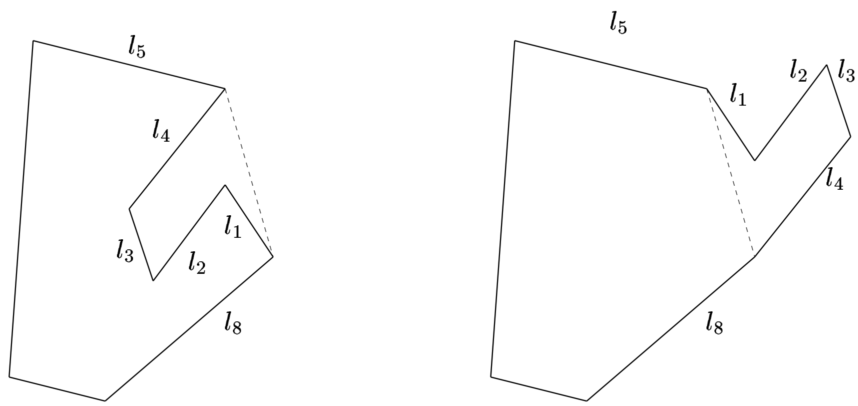

Figure 1: An example of a flipturn applied to pocket $\{l_1, l_2, l_3, l_4\}$. The convex hull is indicated by a dashed line.

2 A polygon that needs many flipturns

The crucial idea to achieve a lower bound of $\approx n^2/4$ flipturns is to construct a polygon where convexifying equals sorting by slopes. We choose the slopes and the length of the links of the polygon in such a way that any flipturn only exchanges two adjacent links of the polygon. By starting the polygon in a suitably shuffled sequence of links, one can show that we need roughly $n^2/4$ flipturns.

More precisely, our polygon is defined for any integer $k \geq 1$, and consists of $2k$ links (which will encode the sequence of slopes to be sorted) as well as two additional links a and b (which serve to complete the polygon). The first $2k$ links will be denoted (in clockwise order) as $l_k, l_{k-1}, l_{k+1}, l_{k-2}, \ldots, l_{2k-2}, l_1, l_{2k-1}, l_0$. Link l_j, for $j = 0, \ldots, 2k-1$, has slope j and extends 2^j units in x-direction, hence $j2^j$ units in y-direction. Link a has a slightly negative slope (say, $-\varepsilon$ for some $\varepsilon > 0$), and link b is vertical. This polygon is illustrated in Figure 2.

The selection rule that leads to a "bad" sequence of flipturns is very simple: we always flip the first pocket after link b (in clockwise order). As we will prove formally below, the follow happens:

- In the first flipturn, we flip the pocket defined by l_{k-1} and l_k, thus exchange l_k and l_{k-1}.
- In the next flipturn, we flip the pocket defined by l_{k-2} and l_{k+1}. In the next flipturn, we flip the pocket defined by l_{k-2} and l_k. In the next flipturn, we flip the pocket defined by l_{k-2} and l_{k-1}. Thus, we have three flips that deal with segment l_{k-2}.
- The next five flipturns all deal with segment l_{k-3}. More precisely, these five flipturns exchange l_{k-3} (in this order) with $l_{k+2}, l_{k+1}, l_k, l_{k-1}, l_{k-2}$.
- This continues. More precisely, for $i = 1, \ldots, k$ we deal with segment l_{k-i} during $2i-1$ flipturns, namely, one flipturn for each exchange with $l_{k+i-1}, l_{k+i-2}, \ldots, l_{k-i+2}, l_{k-i+1}$.

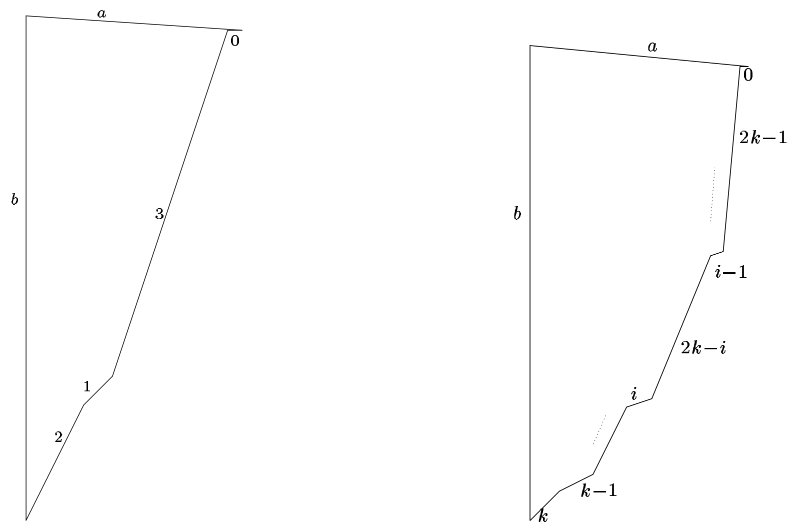

Figure 2: The polygon for $k = 2$, and the general construction (not to scale). Numbers denote slopes.

- The total number of flipturns therefore is at least

$$\sum_{i=1}^{k}(2i-1) = k^2 = (n-2)^2/4.$$

Before we prove this claim, we need an observation regarding slopes.

Claim 1 *Let $0 \leq \beta < \alpha \leq 2k-1$ be two integers. If we attach segment l_α to segment l_β, then the line through the free endpoints of these two segments has slope in the interval $(\alpha - 1, \alpha)$.*

Proof: Since we know the slopes and the lengths of these two segments, we can compute the slope of the line easily; it is

$$\frac{\alpha 2^\alpha + \beta 2^\beta}{2^\alpha + 2^\beta} = \frac{\alpha 2^{\alpha-\beta} + \beta}{2^{\alpha-\beta} + 1}$$

Clearly, this slope is strictly less than α, because $\beta < \alpha$. Also observe that in order for this slope to be strictly greater than $\alpha - 1$, we must have

$$\alpha 2^{\alpha-\beta} + \beta > \alpha 2^{\alpha-\beta} + \alpha - 2^{\alpha-\beta} - 1.$$

This holds, because $2^x > x - 1$ for all $x \geq 0$. □

To prove what flipturns are happening, we analyze the state of the polygon after each flipturn. In essence, the sequence of links l_j is split into two parts: the first part (which contains $l_{k-i+1}, \ldots, l_{k+i-1}$ for some suitable value of i) is sorted by slope, while the second part (which contains $l_0, \ldots, l_{k-i-1}$ and $l_{k+i}, \ldots, l_{2k-1}$) is left as it was originally. Link l_{k-i} is somewhere in the first part, and in fact, "walks" from after the last of these links (l_{k+i-1}) to before the first of these links (l_{k-i+1}) with each flipturn. This is illustrated in Figure 3. The precise statement is as follows:

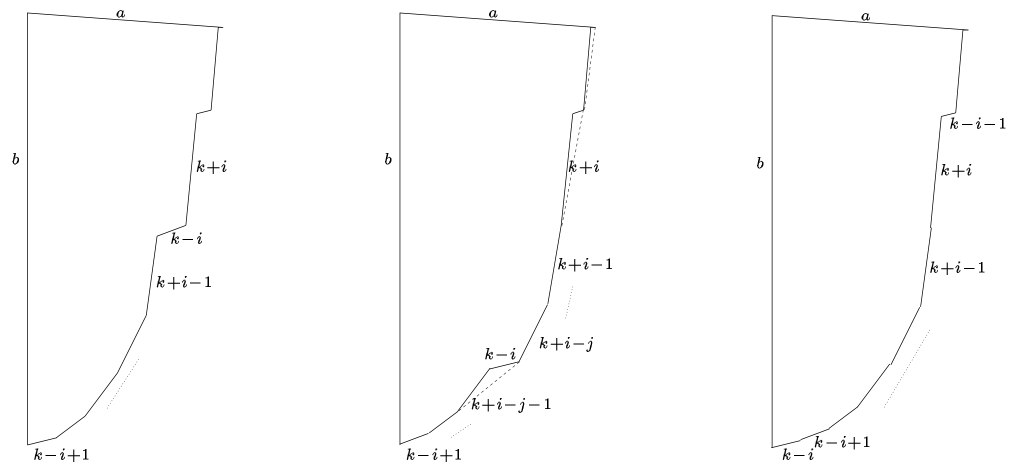

Figure 3: Link l_{k-i} "walks" from after link l_{k+i-1} (before $j=0$) to before l_{k-i+1} (after $j=2i-2$) with $2i-1$ flipturns. In the middle figure (which describes the configuration of the lemma) we also indicate the convex hull with dashed segments.

Claim 2 *Assume we have done l flipturns, $0 \le l \le k^2$. Let $i \ge 1$ be the maximal integer such that $(i-1)^2 \le l$, and let $j = l-(i-1)^2$. Then the polygon has the following form:*

$$l_{k-i+1}, l_{k-i+2}, \ldots, l_{k+i-j-2}, \quad l_{k+i-j-1}, l_{k-i}, \quad l_{k+i-j}, \ldots, l_{k+i-2}, l_{k+i-1},$$

$$l_{k+i}, l_{k-i-1}, \quad l_{k+i+1}, l_{k-i-2}, \quad \ldots, \quad l_{2k-2}, l_1, \quad l_{2k-1}, l_0, \quad a, b.$$

Proof: We prove this claim by induction on l. Assume first that $l=0$, so we have not done any flipturn yet. For $l=0$ we have $i=1$ and $j=0$. The claim states that the polygon has the form

$$l_k, l_{k-1}, l_{k+1}, l_{k-2}, \ldots, l_{2k-2}, l_1, l_{2k-1}, l_0, a, b,$$

which is exactly the initial configuration of the polygon.

Now assume that the claim holds after we have done $l \ge 0$ flipturns. For any segment s that does not have infinite slope, define $p_l(s)$ to be the left endpoint of s and $p_r(s)$ to be the right endpoint of s. We claim that the convex hull of the polygon after doing the lth flipturn consists of the following line segments:

$$l_{k-i+1}, l_{k-i+2}, \ldots, l_{k+i-j-2}, \quad p_l(l_{k+i-j-1})p_r(l_{k-i}), \quad l_{k+i-j}, \ldots, l_{k+i-2}, l_{k+i-1},$$

$$p_l(l_{k+i})p_r(l_{k-i-1}), \quad p_l(l_{k+i+1})p_r(l_{k-i-2}), \quad \ldots, \quad p_l(l_{2k-2})p_r(l_1), \quad p_l(l_{2k-1})p_r(l_0), \quad a, b.$$

See also the middle picture of Figure 3. By Claim 1 the slopes of these line segments are

$$k-i+1, k-i+2, \ldots, k+i-j-2, \quad \in (k+i-j-2, k+i-j-1), \quad k+i-j, \ldots, k+i-2, k+i-1,$$

$$\in (k+i-1, k+i), \quad \in (k+i, k+i+1), \quad \ldots, \quad \in (2k-3, 2k-2), \quad \in (2k-2, 2k-1), \quad -\varepsilon, \infty.$$

Thus, the slopes are strictly increasing (except for segments a and b) and these segments form a convex polygon.

It follows that the clockwise first pocket after link b is the pocket formed by links $l_{k+i-j-1}$ and l_{k-i}. Doing a flipturn on this pocket will exchange the order of the two links.

Now, if $j < 2i-2$, then $l' = l+1 = (i-1)^2 + j + 1 < (i-1)^2 + 2i - 2 + 1 = i^2$ is not a perfect square. Thus we have $i' = i$ and $j' = j' + 1$ (where primed numbers denote the number for $l' = l+1$). One verifies easily that the new order of links around the polygon is exactly as stated, because all that has happened is that link l_{k-i} has moved one link further towards link b.

If $j = 2i-2$, then $k+i-j+1 = k-i+1$, thus after the $(l+1)$st flipturn, link l_{k-i} is the first link after b. But also, if $j = 2i-2$, we have $l' = l+1 = (i-1)^2+j+1 = (i-1)^2+2i-2+1 = i^2$, thus $i' = i+1$ and $j' = 0$. Again one verifies that the new order of links around the polygon is exactly as stated. □

In particular, this claim implies that after $< k^2$ flipturns the predecessor of l_0 is some l_j with $j > 0$, and therefore l_0 and its predecessor form a pocket. Hence the polygon is not convex until at least k^2 flipturns have been done. By $k = (n-2)/2$, we have the following theorem:

Theorem 3 *There exists a polygon with n links such that for some bad selection of flipturns, we need at least $(n-2)^2/4$ flipturns to convexify the polygon.*

3 A smaller polygon that needs many flipturns

The above polygon that needs $\approx n^2/4$ flipturns has one major drawback: the link lengths are exponential in the number of links. Thus, if we disallow scaling (for example by demanding that all vertices of the polygon are placed on grid-points), then the polygon has exponentially big x-coordinates and y-coordinates. Thus in order to store this polygon, one needs $\Omega(n)$ bits per link and $\Omega(n^2)$ bits total.

To overcome this problem, we now give another construction of a polygon that needs many flipturns. This polygon needs only $\approx n^2/8$ flipturns, but in exchange, the link-lengths are at most quadratic in the number of links. Hence the polygon has polynomially big x-coordinates and y-coordinates.

The idea for this construction is exactly the same as for the previous polygon. However, we now make half of the links have slope 0 and length 1; one can then show that it suffices to make the other links shorter.

More precisely, the second polygon is defined for any integer $k \geq 1$, and consists of $2k+2$ links. The first $2k$ links will be denoted (in clockwise order) as $l_1, s_1, l_2, s_2, \ldots, l_k, s_k$. Link l_j, for $j = 1, \ldots, k$, has slope j and extends j units in x-direction, hence j^2 units in y-direction. Link s_j, for $j = 1, \ldots, k$, is horizontal and has length 1. The last two links a and b are as before, i.e., link a has a slightly negative slope $-\varepsilon$ and link b is vertical. This polygon is illustrated in Figure 4.

We prove first the equivalent of Claim 1.

Claim 4 *Let $0 \leq \alpha, \beta \leq k$ be two integers. If we attach segment l_α to segment s_β, then the line through the free endpoints of these two segments has slope in the interval $(\alpha - 1, \alpha)$.*

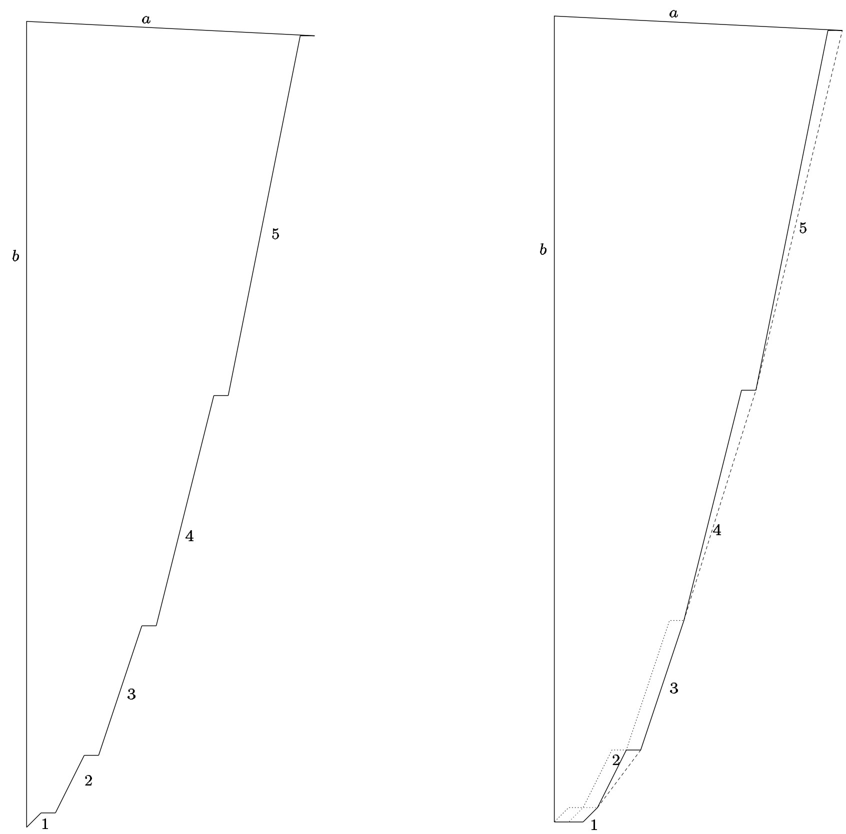

Figure 4: The construction of a smaller polygon that needs many flipturns for $k = 5$. We also show the polygon after four flipturns have been executed, and indicate its convex hull with dashed lines.

Proof: Since we know the slopes and the lengths of these two segments, we can compute the slope of the line easily; it is $\alpha^2/(\alpha+1)$, which is clearly $< \alpha$ and $> \alpha - 1$. □

The equivalent of Claim 2 is as follows:

Claim 5 *Assume we have done l flipturns, $0 \leq l \leq k(k-1)/2$. Let $i \geq 1$ be the maximal integer such that $i(i-1)/2 \leq l$, and let $j = l - i(i-1)/2$. Then the polygon has the following form:*

$$s_1, s_2, \ldots, s_{i-1}, l_1, l_2, \ldots, l_{i-j-1}, \quad l_{i-j}, s_i, \quad l_{i-j+1}, \ldots, l_i, \quad l_{i+1}, s_{i+1}, l_{i+2}, s_{i+2}, \ldots, l_k, s_k, a, b.$$

Proof: The proof is near identical to the proof of Claim 2. For $l = 0$, we have $i = 1$ and $j = 0$, and the desired configuration is exactly the original configuration. Assume we have finished $l \geq 0$ flipturns and the configuration is as desired. Using Claim 4, it follows that the first pocket after b is l_{i-j}, s_i, and the next flipturn will exchange l_{i-j} and s_i. Distinguishing cases by $j < i - 1$ and $j = i - 1$, one obtains the result. Details are left to the reader. □

Thus, this polygon cannot be convex before we have done at least $k(k-1)/2$ flipturns, because otherwise s_k and its predecessor form a pocket. By $k = (n-2)/2$, this implies the following theorem:

Theorem 6 *There exists a polygon with* n *links such that for some bad selection of flipturns, we need at least* $(n-2)(n-4)/8$ *flipturns to convexify the polygon. Furthermore, all link-lengths are polynomial in* n.

4 Conclusion

In this note, we have given two constructions of polygons to provide a lower bound on the number of flipturns needed to convexify a polygon. The first construction needs – with a bad sequence of flipturns – at least $\approx n^2/4$ flipturns, while the second construction needs – with a bad sequence of flipturns – at least $\approx n^2/8$ flipturns and has polynomial link-length.

Unfortunately, both constructed polygons can be convexified with only $O(n)$ flipturns if a good sequence of flipturns is chosen. This leaves the obvious open problem: given a polygon, is there always a sequence of $O(n)$ flipturns that convexifies the polygon?

Another interesting question is whether we could restrict link-lengths to be constant? That is, is there a polygon that needs $\Omega(n^2)$ flipturns, and such that, if all vertices are placed on grid-points, the link-lengths are constant?

References

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