Simultaneous Edge Flips for Convex Subdivisions

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Abstract

It has been shown that one triangulation of a set of points can be converted to any other triangulation of the same set of points by a sequence of *edge flip* operations. In this paper we consider a tesselation of a set of points consisting of convex cells, a *convex subdivision*, and explore the notion of flipping edges from one convex subdivision of the points to another, where both subdivisions use the same number of edges. It is easy to construct examples of a convex subdivision where no single edge can be flipped so that the convexity of all cells of the subdivision is maintained. At the CCCG in 2003 Ferran Hurtado asked whether there exists a convex subdivision for which the size of the minimal simultaneous edge flip is linear with respect to the number of edges. In the paper we construct such a subdivision.

1 Introduction

Given a triangulation of a set of points we define an *edge flip* operation as replacing one diagonal of a convex quadrilateral with the other. The edge flip operation is the basis of a simple algorithm devised by Lawson [6] to convert an arbitrary triangulation of a set of points to a Delauney triangulation. In the Lawson algorithm a diagonal is replaced if it is the chord of a circle containing the fourth point of the convex quadrilateral. This operation can be repeated so that eventually one is left with a triangulation where all triangles are circumscribed by empty circles. One requisite for this algorithm to succeed is the fact that one can always find a sequence of flips that transforms an arbitrary triangulation into a Delaunay triangulation. This implies that one can flip between any two triangulations of a point set. A comprehensive survey of algorithms for computing Delaunay and other triangulations can be found in the book by Sugihara [8].

Hurtado, Noy, and Urrutia [5] consider the question of how far apart two triangulations can be, where the distance between them is the minimum number of flips needed to transform one to the other. In their paper Hurtado et. al. show that $\Theta(n^2)$ flips are sometimes necessary, and always sufficient. Subsequently, Galtier et. al. [2] generalize the notion of the edge flip operation by allowing simultaneous edge flips. A *simultaneous edge flip* allows flipping groups of edges in parallel, with the provision that no two of the edges can be the sides of the same triangle. Galtier et. al. then consider the distance between a pair triangulations of a point set as the number of simultaneous flips needed to transform one triangulation into the other. Of course in this measure a simultaneous flip is counted as one operation independent of the number of edges that get flipped simultaneously. Galtier et. al. show that $\Theta(n)$ simultaneous flips are sometimes necessary and always sufficient to get from one triangulation to another.

Let a *convex subdivision* of a set of points P in the plane be any tesselation of the points into convex cells. A convex subdivision of a point set is an attractive alternative to a triangulation of a point set, because it uses fewer edges and requires less storage space. A polynomial time algorithm to compute a minimum convex subdivision, that is, a convex subdivision using the fewest number of elements, is given for certain special cases by Fevens et. al. [1]. In general, the complexity of computing a minimum convex subdivision is not known. Extremal values for the number of cells in a minimal convex subdivision of a set of points has a long history. A recent paper by Neumann-Lara et. al [7] shows that every set of points has a minimal convex subdivision using no more than $(3n-6)/2$ cells.

Let $A(P)$ be a convex subdivision of a set of points P. Notice that if no three points in P are colinear and if any point of P not on the convex hull of P has degree three in $A(P)$, then $A(P)$ is a minimal subdivision and given any line h through a point p of P not on the convex hull of P, there is an edge of $A(P)$ incident on p on each side of h. Also if $A(P)$ is a convex subdivision without three colinear points, then any point of P not on the convex hull of P has degree at least three. A *simultaneous edge flip* of size k in $A(P)$ replaces k edges of $A(P)$ with k new edges, such that the result is again a convex subdivision. Note that in this scenario we no longer require that the flipped edges be independent in any way. Given a convex subdivision $A(P)$ a *minimal simultaneous edge flip* is a simultaneous edge flip of $A(P)$ of the smallest size. At the Canadian Conference on Computational Geometry held in Halifax in 2003 Ferran Hurtado asked whether there were convex subdivisions $A(P)$ and $B(P)$ such that $A(P)$ could only be transformed into $B(P)$ by using minimal simultaneous edge flips of size $\Theta(n)$ [4]. In this paper we answer this question in the affirmative by providing a construction of a convex subdivision with $3n$ points such that a minimal simultaneous edge flip uses n edges.

2 Construction

Lemma 2.1 *For all n with $n > 4$, there is a set of $3n$ points for which there exist a minimal simultaneous edge flip of size n.*

Proof: Consider the following set of $3n$ points P, which is illustrated in Figure 1 with $n = 12$. Place n points at the corners of a regular n-gon centered at the origin with one vertex at $(x, y) = (1, 0)$. Let C_0 be this set of n points and number the points by $0, 1, 2, \ldots, n-1$ in counter clockwise direction starting with the point at $(x, y) = (1, 0)$. Draw a regular n-gon centered at the origin, such that each vertex has distance r_1 to the origin, where $r_1 = \cos(\pi/n) - \varepsilon_1$ for some sufficiently small positive value of ε_1, with one vertex at $(x, y) = (\cos(\pi/n)r_1, \sin(\pi/n)r_1)$. Place n points at its vertices and denote this set of points by C_1. Number the points in C_1 by $n, n+1, n+2, \ldots, 2n-1$, in counter clockwise direction starting with the point at $(x, y) = (\cos(\pi/n)r_1, \sin(\pi/n)r_1)$. Construct a regular n-gon centered at the origin, such that each point has distance r_2 to the origin, where

$$r_2 = \frac{r_1 \cos\frac{2\pi}{n}}{\cos\frac{\pi}{n}} - \varepsilon_2$$

for some sufficiently small positive value of ε_2 with one vertex at $(x, y) = (r_2, 0)$. Place n points at its vertices and denote this set of points by C_2. Number the points in C_2 by $2n, 2n+1, 2n+2, \ldots, 3n-1$ in counter clockwise direction starting with the point at $(x, y) = (r_2, 0)$. The value of ε_1 should be small enough such that only point 1 lies above the line through points 0 and n. The value of ε_2 should be small enough such that only points $\{0, 1, n, n+1, 2n+1\}$ lie above the line through points $2n-1$ and $2n$. Finally we move each point a little so that no 3 points are colinear.

We now construct a convex decomposition of P, called $D_0(P)$. We first add the edges $(i, i+1 \bmod n)$ for $0 \le i < n$ as well as edges $(i, n+i)$ and $(n+i, i+1 \bmod n)$ for $0 \le i < n$. Then we add edges $(n+i, 2n+i)$ and $(2n+i, 2n+(i+1 \bmod n))$ for $0 \le i < n$. Since each point in $D_0(P)$ not on the convex hull of P has degree three and since no three points in P are colinear, $D_0(P)$ is a minimal convex subdivision of P. This subdivision is shown in Figure 1.

Notice that we can construct another convex subdivision, called $D_1(P)$, with the same number of edges as $D_0(P)$, by flipping the edges between C_1 and C_2, i.e. the edges $(n+i, 2n+i)$ are replaced by the edges $(n+i, 2n+(i+1 \bmod n))$. We will show that $D_0(P)$ and $D_1(P)$ are the only two convex subdivisions with this number of edges. That implies that in order to transform $D_0(P)$ into $D_1(P)$ we need to flip all n edges between C_1 and C_2 simultaneously. That would prove the lemma.

Assume we want to construct an arbitrary convex subdivision $D_2(P)$ with the same number of edges as $D_0(P)$. First note that $D_2(P)$ must contain the edges of $D_0(P)$ between points in $C_0 \cup C_1$. Observe that at this moment in $D_2(P)$ the points in C_0 have degree four, the points in C_1 have degree two and the points in C_2 have degree zero. Since the points in C_0 have degree four in $D_0(P)$ and since $D_2(P)$ is a minimal convex subdivision, if follows that all points in $C_1 \cup C_2$ will have to be of degree three in $D_2(P)$. We say that a point in C_0 of degree four and a point in $C_1 \cup C_2$ of degree three is full.

Suppose that in $D_2(P)$ there is an edge $(n, n+1)$. Consider the line h through $2n+1$ such that only points $\{0, 1, 2, n, n+1\}$ lie on one side of h. The point $2n+1$ has to have an incident edge above h. However all points above this line are already full. So $D_2(P)$ has no edge $(n, n+1)$ and by the same argument has no edges $(n+i, n+(i+1 \bmod n))$ for $0 \leq i < n$.

Consider a line h through $2n+1$ such that only points $\{0, 1, 2, n, n+1\}$ lie on one side of h. The point $2n+1$ has to have an incident edge above h. Points $0, 1, 2$ are already full. So either there is an edge $(n, 2n+1)$ or an edge $(n+1, 2n+1)$ or both. Assume we have edge $(n+1, 2n+1)$. Consider the line h through $n+2$ and $2n+2$. Since $2n+1$ is the only non-full point above this line visible from $2n+2$, $D_2(P)$ contains the edge $(2n+1, 2n+2)$. Consider the line h through $2n+1$ and $2n+2$. Since $n+2$ is the only non-full point above this line visible from $2n+2$, we need to have edge $(n+2, 2n+2)$. Continuing this argument, we arrive at $D_2(P) = D_0(P)$.

If we start with edge $(n, 2n+1)$ rather than $(n+1, 2n+1)$ we get $D_2(P) = D_1(P)$. If we start with both edges $(n, 2n+1)$ and $(n+1, 2n+1)$ the result is not a minimal convex subdivision of P. So $D_0(P)$ and $D_1(P)$ are the only two minimal convex subdivisions of P. □

3 Discussion

One can construct a flip graph of triangulations of a set of points P, where nodes of the graph are triangulations and two nodes are adjacent if the triangulations differ by a single flip. In this scenario we can say that the graph of triangulations for P is connected and it has diameter $O(n^2)$. For the case of convex subdivisions, if we restrict ourselves to constant size simultaneous flips a similarly constructed flip graph may not be connected. We have given a construction of a convex subdivision that has a minimal simultaneous edge flip of size $\Theta(n)$. The construction yields only two distinct convex subdivisions with the same number of edges. Thus for our example the diameter of the flip graph is a constant if we allow minimal simultaneous flips of arbitrary size. It would be interesting to determine whether there is a set P with a convex subdivision that has a large minimal simultaneous edge flip size, and also results in a flip graph with large diameter.

Huemer et. al [3] study a graph of convex subdivisions of a convex set. When the points are in convex position then one can always go from a decomposition to another using a single flip. Huemer et. al. show that the flip graph obtained is Hamiltonian. They also consider a variant of this graph where edges may be removed without replacement or added without deletion, and show that it too is Hamiltonian. Thus it would be interesting to explore the combinatorial properties of the flip graphs when using minimal simultaneous edge flips of convex subdivisions.

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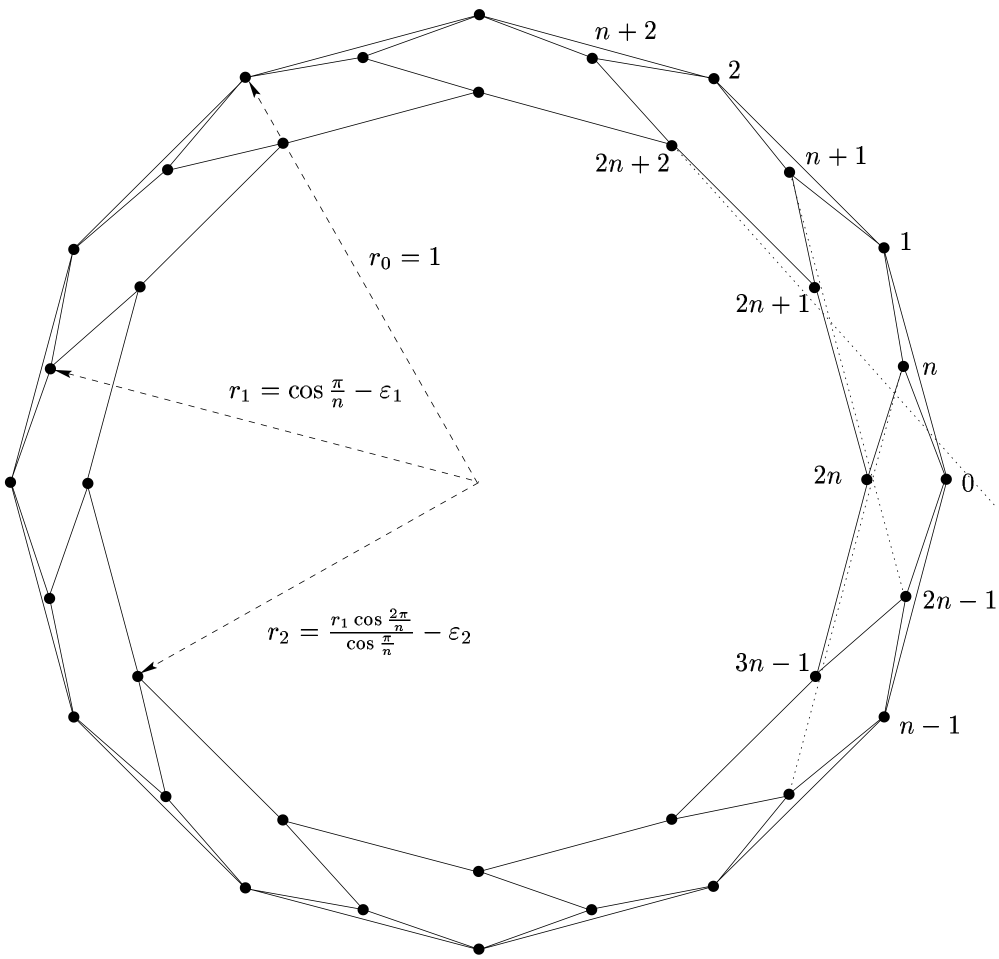

Figure 1: Minimal decomposition of P.

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