Hinged Dissection of Polypolyhedra

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Abstract

This paper presents a general family of 3D hinged dissections for *polypolyhedra*, i.e., connected 3D solids formed by joining several rigid copies of the same polyhedron along identical faces. (Such joinings are possible only for reflectionally symmetric faces.) Each hinged dissection consists of a linear number of solid polyhedral pieces hinged along their edges to form a flexible closed chain (cycle). For each base polyhedron P and each positive integer n, a single hinged dissection has folded configurations corresponding to all possible polypolyhedra formed by joining n copies of the polyhedron P. In particular, these results settle the open problem posed in [DDE+] about the special case of polycubes (where P is a cube) and extend analogous results from 2D [DDE+], Along the way, we present hinged dissections for polyplatonics (where P is a platonic solid) that are particularly efficient: among a particular type of hinged dissection, they use the fewest possible pieces.

1   Introduction

A *dissection* of two figures (solid 2D or 3D shapes, e.g., polygons or polyhedra) is a way to cut the first figure into finitely many (compact) pieces and to rigidly move those pieces to form the second figure. Dissections have been studied extensively, particularly in 2D [Fre97, Lin72, KKU00]. It is well-known that any two polygons of the same area have a dissection [Bol78, Fre97, Low14]. Thus, in this context, the main interest is in finding the dissection of the polygons that uses the fewest possible pieces. On the other hand, not every two polyhedra of the same volume have a dissection; for example, there is no dissection of a regular tetrahedron and an equal-volume cube [Bol78]. This result was a solution to Hilbert's Third Problem [Bol78].

Figure 1: Hinged dissection of square and equilateral triangle, described by Dudeney [Dud02].

A *hinged dissection* of two figures is a dissection in which the pieces are hinged together at points (in 2D or 3D) or along edges (in 3D), and there is a motion between the two figures that adheres to the hinging, keeping the hinge connections between pieces intact. While a few hinged dissections such as the one in Figure 1 are quite old [Dud02], hinged dissections have received most of their study in the last few years [AN98, DDE+, Epp01, Fre02]. It remains open whether every two polygons of the same area have a hinged dissection, or whether every two polyhedra that have a dissection also have a hinged dissection.
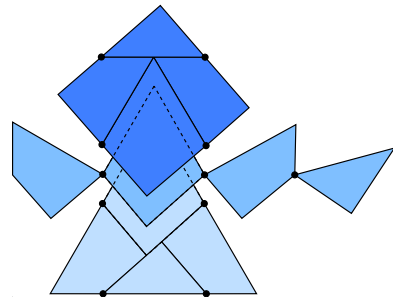
In this paper we develop a broad family of 3D hinged dissections for a class of polyhedra called polypolyhedra. For a polyhedron P with labeled faces, a *polypolyhedron of type P* is an interior-connected non-self-intersecting solid formed by joining several rigid copies of P wholly along identically labeled faces. See Figure 2. These joinings must perfectly match two opposite orientations of the same face of P, so joinings can occur only along faces with reflectional symmetry. We call P the *base polyhedron*. If a polypolyhedron consists of n rigid copies of P, we call it an *n-polyhedron of type P*. Examples of polypolyhedra include *polycubes* (where P

Figure 2: Joining two rigid copies of a tetrahedron. The face of joining is reflectionally symmetric.

is a cube) or more generally *polyplatonics* (where P is any fixed platonic solid); in any of these cases, any pair of faces can be joined because of the regular symmetry of the platonic solids. See Figure 3 for some examples of polycubes.

For every polyhedron P and positive integer n, we develop one hinged dissection that folds into all (exponentially many) n-polyhedra of type P. This result is superior to having one hinged dissection between every pair of n-polyhedra of type P. The number of pieces in the hinged dissection is linear in n and the combinatorial complexity of P. For polyplatonics, we give particularly efficient hinged dissections, tuning the number of pieces to the minimum possible among a natural class of "regular" hinged dissections of polypolyhedra. For polyparallelpipeds (where P is any fixed parallelpiped), we give hinged dissections in which every piece is a scaled copy of P. All of our hinged dissections are hinged along

Figure 3: Two polycubes of order 8, which have a 24-piece edge-hinged dissection by our results.

edges and form a cyclic chain of pieces, which can be broken into a linear chain of pieces.

Our solution combines several techniques to obtain increasingly more general families of hinged dissections. We reduce the problem of finding a hinged dissection of polypolyhedra of type P to finding a hinged dissection of P that has "exposed hinges" at certain locations on its surface. We find the first such hinged dissection for every platonic solid, exploiting that such a solid is star-shaped and has a Hamiltonian cycle on its faces. Then we relax the star-shaped constraint, generalizing P to be any solid with a Hamiltonian cycle on its faces, using a more general refinement scheme based on the straight skeleton. Then we relax the Hamiltonicity constraint by using a Hamiltonian refinement scheme. Finally, we show how faces with more than a single reflectional symmetry can be joined even when their labeled rotations are not equal. This step uses a general "twister" gadget, a hinged dissection that can rotate by any angle that is a multiple of $360°/k$ for fixed k.

Our results generalize analogous results about hinged dissections of "polyforms" in 2D [DDE$^+$]. For a polygon P with labeled edges, a *polyform of type P* is an interior-connected non-self-intersecting planar region formed by joining several rigid copies of P wholly along identically labeled edges. In particular, polyforms include polyominoes (where P is a square) and polyiamonds (where P is an equilateral triangle). In 2D, edges are always reflectionally symmetric (about their midpoint), so a polyform can join any pair of identically labeled edges. For any polygon P and positive integer n, [DDE$^+$] develops a single vertex-hinged dissection that folds into all n-forms of type P. The same paper asks whether analogous dissections exist in 3D, in particular for polycubes; we solve this open problem, building on the general inductive approach of [DDE$^+$].

We do not know whether our hinged dissections can be folded from one configuration to another without self-intersection. (The same is true of most previous theoretical work in hinged
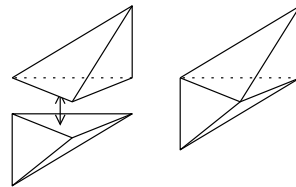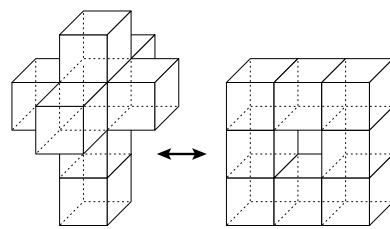
dissections [AN98, DDE$^+$, Epp01].) However, we prove the existence of such motions for the most complicated gadget, the twister.

2 Polyplatonics

In this section we demonstrate our approach for constructing a hinged dissection of polypolyhedra of type P in the special case that P is a platonic solid. Although several of the details change in more general settings in later sections, the overall approach remains the same.

First, we find a suitable hinged dissection of the base polyhedron P. The exact constraints on this dissection vary, but two necessary properties are that the hinged dissection must be (1) cyclic, forming a closed chain of pieces, and (2) *exposed* in the sense that, for every face of P, there is a hinge in H that lies on the face (either interior to the face or on its boundary). For platonic solids, these hinges will be edges of the polyhedron. Second, we repeat n copies of this hinged dissection of P, sliced together into one long closed chain. Finally, we prove that this new hinged dissection can fold into all n-polyhedra of type P, by induction on n.

2.1 Exposed Cyclic Hinged Dissections of Platonic Solids

We construct an exposed cyclic hinged dissection of any platonic solid as follows. First we carve the platonic solid into a cycle of face-based pyramids with the platonic solid's centroid as the apex. Thus, a refined tetrahedron consists of four triangle-based pyramids (irregular tetrahedra); a refined cube consists of six square-based pyramids; a refined octahedron consists of eight triangle-based pyramids; a refined dodecahedron consists of twelve pentagon-based pyramids; and a refined

Figure 4: Carving a regular tetrahedron into four face-based pyramids with apexes at the centroid.

icosahedron consists of twenty triangle-based pyramids. Every platonic solid has a Hamiltonian path on its faces.[1] Consequently, the pieces in the refinement can be hinged together in a cycle, following the Hamiltonian path on the faces. Figure 5 shows unfoldings of these hinged dissections, in particular illustrating the Hamiltonian cycle.
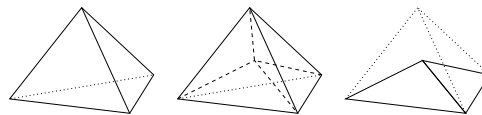
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[1]This fact was probably known to Hamilton, who in the 1850s invented the Icosian Game in which the goal is to construct a Hamiltonian path on the faces of an icosahedron [Gar59].
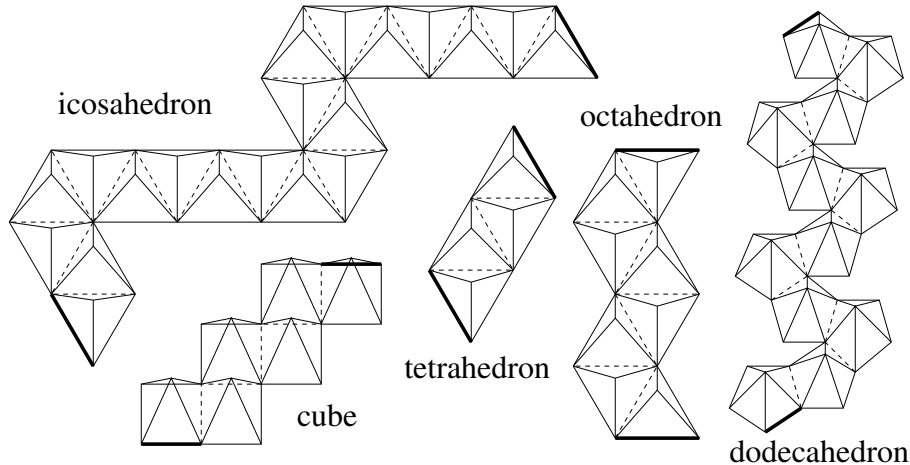
Figure 5: Unfolded exposed cyclic hinged dissections of the platonic solids. The bold lines indicate a pair of edges that are joined by a hinge but have been separated in this figure to permit unfolding. The dashed lines denote all other hinges between pieces. In the unfolding, the bases of all of the pyramid pieces lie on a plane, and the apexes lie above that plane (closer to the viewer).

Because there is a hinge dual to every edge in the Hamiltonian path on the faces, every face of the platonic solid has exactly two hinges. Therefore, the hinged dissection is exposed. Even more, we can merge adjacent pairs of pyramids along a face, halving the number of pieces, and still have exactly one hinge per face of the platonic solid. Now two faces share every hinge, but still the hinged dissection is exposed because every face has a hinge along its boundary. Thus we have proved

Theorem 1 *The platonic solid with f faces has an exposed cyclic hinged dissection of $f/2$ pieces in which every hinge is an edge of the platonic solid.*

These exposed hinged dissections have the fewest possible pieces, subject to the exposure constraint, because a hinge can simultaneously satisfy at most two faces of the original polyhedron.

2.2   Inductive Hinged Dissection

Next we show how to build a hinged dissection of all n-platonics of type P based on a repeatable hinged dissection of a platonic solid P. The hinged dissection is essentially n repetitions of the exposed hinged dissection from the previous section. Specifically, the *nth repetition* of a cyclic hinged dissection is the result of cutting the cyclic hinged dissection at an arbitrary hinge to form an open chain, repeating this open chain n times, and then reconnecting the ends to restore a closed chain. Thus, if there are k pieces $H_1, H_2, \ldots, H_k$ connected in that order (and cyclically) in a cyclic hinged dissection, then the nth repetition consists of nk pieces $H_1, \ldots, H_k, H_1, \ldots, H_k, \ldots \ldots H_1, \ldots, H_k$ connected in that order (and cyclically). (Although the order $H_1, \ldots, H_k$ depends on where we cut the cyclic order, the resulting nth repetition is independent of this cut.)

We prove that this hinged dissection has the desired foldings by an inductive/incremental construction which relies on the following tool, similar to [DDE+, Proposition 1]:

Lemma 1 *Every n-polyhedron of type P has a copy of P whose removal results in a (connected) $(n-1)$-polyhedron, provided $n > 1$.*

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Proof: The graph of adjacencies between copies of P in an n-polyhedron is a connected graph on n vertices. Any spanning tree of this graph has at least two leaves, and the removal of either leaf leaves the original graph connected. The resulting pruned graph is the adjacency graph of a connected $(n-1)$-polyhedron. □

Reversing the inductive process of this lemma implies that any n-polyhedron of P can be built up by adding one copy of P at a time, yielding a connected 1-, 2-, ..., and $(n-1)$-polyhedron along the way.

Theorem 2 *Given an exposed cyclic hinged dissection of the platonic solid P in which exactly one piece is incident to each face of P, the nth repetition of this hinged dissection can fold into any n-platonic of type P.*

Proof: The proof is by induction. The base case of $n = 1$ is trivial: there is only one 1-platonic of type P, namely P itself. The exposed hinged dissection satisfies all the desired properties.

Consider an n-platonic Q of type P. By Lemma 1, one copy P_1 of P can be removed from Q to produce an $(n-1)$-platonic Q'. By induction, the $(n-1)$st repetition of the exposed hinged dissection can fold into Q'. Also, P_1 itself can be decomposed into an instance of the exposed hinged dissection. Our goal is to merge these two hinged dissections.

Let P_2 denote a copy of P in Q' that shares a face f with P_1. Suppose the exposed cyclic hinged dissection of P consists of pieces $H_1, H_2, \ldots, H_k$ in that order. Let H_i denote the piece in the hinged dissection of P_2 incident to face f. Let h be a hinge incident to f (which must be an edge of f) and thus incident to H_i. Suppose by symmetry that the other piece in Q' incident to hinge h is H_{i+1}.

Then we rotate P_1 so that its piece H_{i+1} is flush against the H_i piece in P_2, along the shared face f between P_1 and P_2. We further rotate P_1 so that the hinge h' between pieces H_i and H_{i+1} in P_1 aligns with the hinge h between pieces H_i and H_{i+1} in P_2. We then replace hinges h and h' with two hinges, one from H_i in P_2 to H_{i+1} in P_1, and the other from H_i in P_1 to H_{i+1} in P_2. The resulting hinged dissection is a single cycle, and every instance of piece H_i hinges to pieces H_{i-1} and H_{i+1}, so the resulting hinged dissection is a folding of the nth repetition of $H_1, H_2, \ldots, H_k$ as desired. □

Corollary 1 *If P is the platonic solid with f faces, then there is an $(nf/2)$-piece cyclic hinged dissection that can fold into all n-platonics of type P.*

3   Generalized Interior Dissection

The proof of hinged dissections for polyplatonics consists of two main parts: (1) the construction of an exposed cyclic hinged dissection of a single platonic solid, with the property that at most one piece is incident to each face, and (2) an inductive argument about the nth repetition. In this section we generalize the first part to any polyhedron with a Hamiltonian cycle on its faces. The second part will remain restrictive until future sections.

3.1   Exposed Cyclic Hinged Dissections of Hamiltonian Polyhedra

The exposed cyclic hinged dissection for platonic solids from Section 2.1 essentially exploited that platonic solids, like all convex polyhedra, are "star-shaped". A polyhedron is *star-shaped* if it has at least one point c in its interior from which the line segment to any point on the polyhedron's

surface remains interior to the polyhedron. Any star-shaped polyhedron can be carved into face-based pyramids with apexes at c. These pyramids can be hinged together cyclically at the edges of the polyhedron crossed by the Hamiltonian cycle that visits each face exactly once.

Dissection of a polyhedron into face-based pyramids with a common apex is possible precisely when the polyhedron is star-shaped. However, it is not hard to obtain a dissection of an arbitrary polyhedron into one piece per face, though the pieces are no longer pyramids. One approach is to use the *straight skeleton* [AAAG95, AA96, EE99, CV02]. The straight skeleton is normally defined as a particular one-dimensional tree structure contained in a given two-dimensional polygon. For our purposes, the relevant property is that the tree structure subdivides the polygon into exactly one region per polygon edge, and only that region is incident to that polygon edge [AAAG95].

The straight skeleton can be generalized to 3D as a decomposition of a given polyhedron into exactly one cell per facet, and only that cell is incident to that facet. We imagine sweeping every facet perpendicularly inwards at the same speed in parallel. Faces change geometry as they are inset by clipping or extending to where they meet adjacent faces. Faces may become disconnected, in which case the sweep continues with each piece, or disappear, in which case the sweep continues without that face. In the end, the entire polyhedron is swept, and the regions swept by individual faces form a partition with the desired property that exactly one region is incident to each facet. Erickson [Eri00] points out that the straight skeleton is no longer well-defined in 3D: there are choices during the offset process that can be resolved multiple ways. However, for our purposes, we just need a single straight skeleton, with an arbitrary decision for each choice, for a suitable decomposition.

As before, the pieces can be hinged together cyclically at the edges of the polyhedron crossed by the Hamiltonian cycle. Thus, for any polyhedron with a Hamiltonian cycle on its n faces, we obtain an n-piece exposed cyclic hinged dissection with the property that each face of the polyhedron is incident to exactly one piece.

3.2 Inductive Hinged Dissection

The second part of the argument is the inductive construction. The key steps here are the two rotations of an added piece P_1. The first rotation ensures that the next piece in the hinging of P_1 (H_{i+1}) is against the piece to which we want to join P_1 (H_i of P_2). The second rotation ensures that the exposed hinges of these two pieces coincide.

These rotations enforce restrictions on what types of polypolyhedra we can build. The first rotation essentially requires that all faces of P "look the same" (in addition to having the same shape): the rotation that brings any face to any other face should result in an identical copy of P (but with faces relabeled). The second rotation requires that all orientations of a face look the same. Unfortunately, these two restrictions force P to be a platonic solid. The goal of the remaining sections is to remove these restrictions, in addition to the restriction that P have a Hamiltonian cycle on its faces.

4 Surface Refinement

In this section we remove two constraints on the base polyhedron P: the requirement that P has a Hamiltonian cycle on its faces, and the requirement that all faces of P look the same. We achieve both of these generalizations by subdividing each face of P by a collection of linear cuts.

First, we divide each reflectionally symmetric face of P along one of its lines of symmetry. Recall that joinings between copies of P are possible only along reflectionally symmetric faces. Now if

we can arrange for these symmetry lines to be hinges in an exposed cyclic hinged dissection of the new polyhedron P', then whenever we attempt to attach a new piece P'_1, we are guaranteed that the two consecutive pieces H_i and H_{i+1} of the hinging that we need to place against each other are in fact the two reflectional halves of the original face. Thus the first rotation in the induction construction does exactly what we want: it brings together the two identically labeled faces of P.

Second, we divide each face of P' so that any spanning tree of the faces in P' translates into a Hamiltonian cycle in the resulting polyhedron P''. This reduction is similar to the Hamiltonian triangulation result of [AHMS96] as well as a refinement for hinged dissection of 2D polyforms [DDE+, Section 6]. We conceptually triangulate each face f of P' using chords (though we do not cut along the edges of that triangulation). Then, for each triangle, we cut from an arbitrarily chosen interior point to the midpoints of the three edges. Figure 6 shows a simple example of this process. For any spanning tree of the faces of P', we can walk around the tree (i.e., follow an Eulerian tour) and produce a Hamiltonian cycle on the faces of P''.
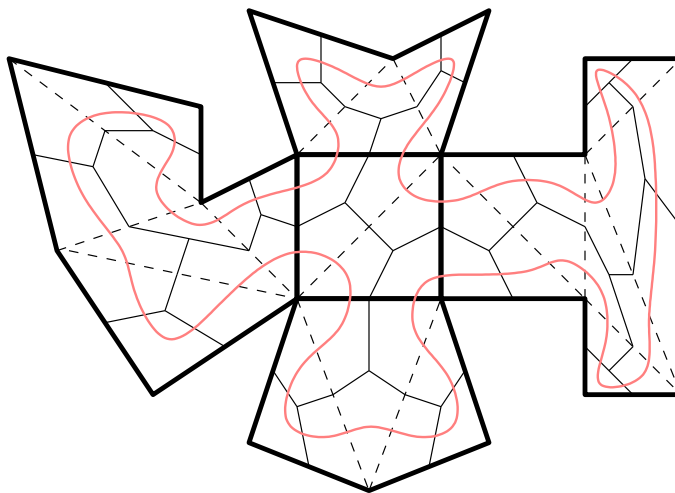

Figure 6: Hamiltonian refinement of five faces in a hypothetical polyhedron, shown here unfolded. Bold lines outline faces. Dashed lines show triangulations and are not cuts. Thin solid lines are cuts. The curve line shows a Hamiltonian cycle induced by the spanning tree of this unfolding.

In particular, we can start from the matching on the faces of P' from the reflectionally symmetric pairing, and choose a spanning tree on the faces of P' that contains this matching. Then the resulting Hamiltonian cycle in P'' crosses a subdivided edge of every line of symmetry. (In fact, the Hamiltonian cycle crosses every subdivided edge of every line of symmetry.) Thus, in the exposed cyclic hinged dissection of the Hamiltonian polyhedron P'', there is an exposed hinge along every line of symmetry. Therefore all joinings between copies of P'' can use these hinges, which means that the first rotation in the induction construction happens automatically from joining along corresponding faces.

5 Mutually Rotated Base Polyhedra: Twisters

The last generalization concerns the second rotation in the inductive construction. If every reflectionally symmetric face has only one line of symmetry, this second rotation is automatic just from making the faces meet geometrically. However, if a face has more than one line of symmetry, the polypolyhedron may require different rotations of the two base polyhedra around their common

face.

To enable these kinds of joinings, we introduce the *twister gadget* shown in Figure 7. This gadget consists of $8k$ cyclically hinged pieces that allow any integer multiple of $1/k$ rotation of one set of pieces with respect to the other pieces. Specifically, we start with two thin prisms with regular k-gon bases, one stacked on top of the other. Then we divide each prism by cutting the projected regular k-gon from the center to every vertex, the midpoint of every edge, and each quarter point between a vertex and an edge midpoint. The resulting pieces are all triangular prisms.
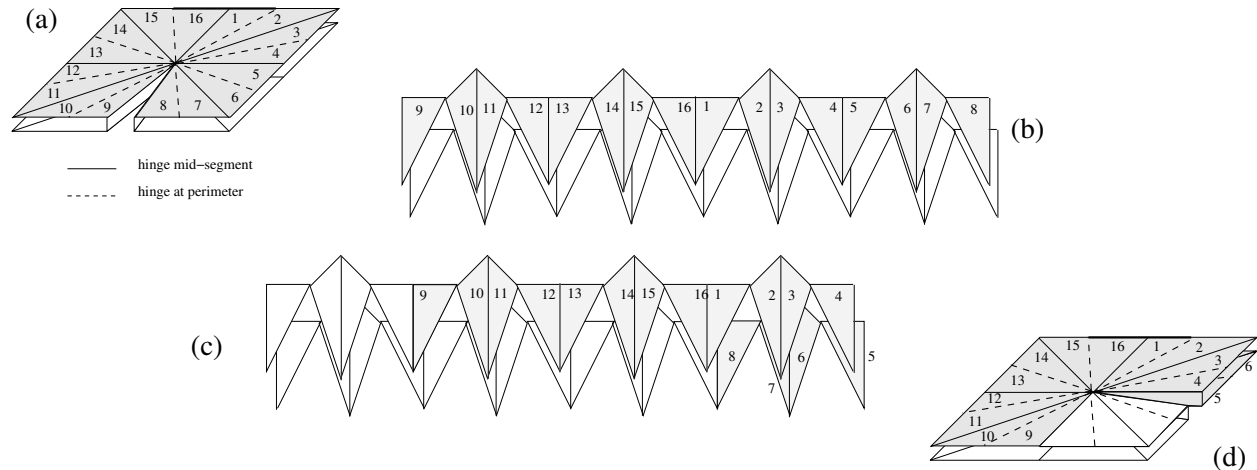

Figure 7: This twister gadget which allows turns of one-eighth of a twist has exactly 32 pieces. Although the pieces look two dimensional, they have thickness (they are prisms). The gaps between pieces 8 and 9 in subfigure (a) and between the top and bottom layers are for visual clarity only; in fact, the two layers are flush. Subfigure (b) shows the result of unfolding the perimeter hinges. This cycle can be rotated some even number of pieces (four as shown in subfigure (c)) using inside hinges. Then the perimeter hinges are used to return to a stacked k-gon as shown in subfigure (d). This figure shows how the gadget can be folded so that the bottom layer rotates while the top layer (particularly piece 9) remains fixed.

We hinge these prisms together as follows. Two hinges connect the top and bottom levels, lying (in projection) along a cut from the center to an edge midpoint. For each remaining cut from the center to an edge midpoint (in projection), and for each cut from the center to a vertex (in projection), there is a hinge connecting the two incident pieces on the "inside" (on the bottom of the top prism and on the top of the bottom prism). For each cut from the center to a quarter point (in projection), there is a hinge connecting the two incident pieces on the perimeter of the regular k-gon.

The perimeter hinges enable the twister to unfold as shown in Figure 7(b). Using the inside hinges, which in this flat configuration are all parallel, the twister can be further unfolded into a 3D ring. Then we can reverse the process, collapsing the 3D ring back down to a flat unfolding like Figure 7(c), and folding it back into the regular k-gon configuration. During this process, we can change which pieces are ultimately on which layer as shown in Figure 7(d).

We embed this twister gadget in each face of the base polyhedron P'' that has k-fold symmetry for $k \geq 3$. More precisely, we carve out of P'' a thin prism with a small regular k-gon base, centered at the symmetry center of the face, and infuse this carved space with a twister gadget. Then we reroute the hinges to connect to a piece of the twister gadget that does not move during a twist operation. The resulting polyhedron P''' can rotate itself about this face while keeping fixed its hinge connection to another copy of P'''. This effects the second rotation in the inductive argument

of hinged dissectability.

This completes our construction of a hinged dissection that folds into all n-polyhedra of type P, for any positive integer n and for any polyhedron P.

6   Self-Similar Hinged Dissections

Figure 8 gives an alternative hinged dissection of a cube that leads to a hinged dissection of n-cubes for any n. The dissection is simple, dividing the cube into a $2 \times 2 \times 2$ array of identical subcubes. The hinged is more complex because of the requirement that every face of the original cube has an exposed hinge. The hinges are always between the midpoint of an original edge to the center of an original face, so they align with adjacent cubes for merging.
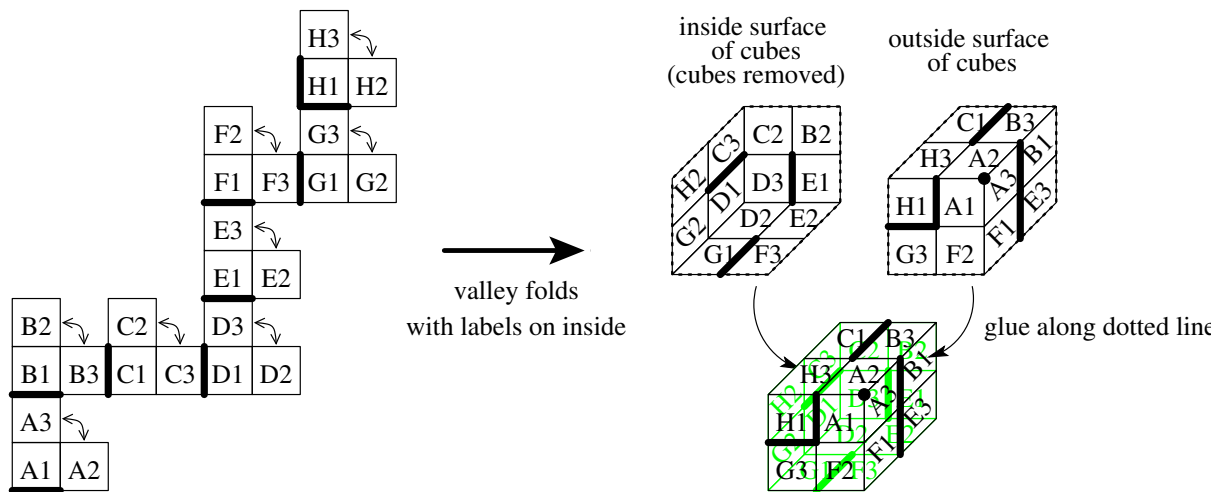

Figure 8: A hinged dissection of a cube into eight cubes. This hinged dissection can be used in place of that in Figure 5; every face has (at least) one hinge. The left-hand side of the figure shows the surface of the cube, and the locations of the eight hinges.

One attractive feature of this hinged dissection, although it uses $8n$ pieces instead of $2n$ pieces as in Corollary 1, is that all the pieces are cubes. We call such a hinged dissection *self-similar*: the pieces are similar to the original shape P. This hinged dissection is clearly the smallest non-trivial self-similar hinged dissection of the cube, and hence is optimal among such dissections. The dissection also applies more generally to a base shape P of any parallelpiped, e.g., any box.

Finding self-similar hinged dissections for other polypolyhedra seems difficult. For example, no other platonic solid can be tiled by smaller copies of itself.

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