6

EXTERIOR VISIBILITY

6.1. INTRODUCTION

Derick Wood and Joseph Malkelvitch independently posed two interesting variants of the original Art Gallery Problem, which Wood dubbed *The Fortress Problem* and *The Prison Yard Problem.* The first asks for the number of guards needed to see the exterior of a polygon, and the second asks for the number needed to see both the exterior and the interior. The first has been satisfactorily solved, but the second remains tantalizingly open.

6.2. FORTRESS PROBLEM

How many vertex guards are needed to see the exterior of a polygon of n vertices? An exterior point y is seen by a guard at vertex z iff the segment zy does not intersect the interior of the polygon.

6.2.1. General Polygons

A convex n-gon establishes that $\lceil n/2 \rceil$ guards are occasionally necessary: a guard is needed on every other vertex. See Fig. 6.1. One might conjecture that in fact placing guards on every other vertex is always sufficient, even for non-convex polygons, but Fig. 6.2 (an example due to Shermer) demonstrates that this simple strategy will not work: placing guards on either the odd or the even indexed vertices leaves a portion of the exterior uncovered. It is, however, not difficult to establish the sufficiency of $\lceil n/2 \rceil$ using a 3-coloring of an exterior triangulation.

THEOREM 6.1 [O'Rourke and Wood 1983]. $\lceil n/2 \rceil$ vertex guards are necessary and sufficient to see the exterior of a polygon of n vertices.

Proof. Triangulate the portion of the plane that is inside of the convex hull but exterior to the polygon. Call the resulting graph of n nodes G''; see Fig.

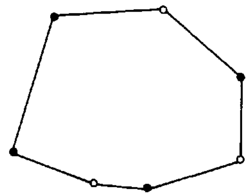

Fig. 6.1. A convex polygon requires $\lceil n/2 \rceil$ vertex guards (solid dots) to cover the exterior.

6.3a. Add an additional node v_∞ to G'' outside the hull and make it adjacent to every node on the hull; call this graph of $n+1$ nodes G' (Fig. 6.3b). Finally, choose some hull vertex x and split it into two vertices x' and x'', apportioning the previous connections from x between x' and x'' so that the graph remains planar, and adding a new arc so that v_∞ is adjacent to both x' and x''. Call the resulting graph of $n+2$ nodes G (Fig. 6.3c).

We claim that this graph is a triangulation graph of a polygon. This can be seen by "opening up" the convex hull at $x' - x''$ and moving v_∞ far enough away to permit all its connections to be straight lines. Since G is a triangulation graph, it can be 3-colored. The least frequently used color, say red, occurs no more than $\lfloor (n+2)/3 \rfloor$ times. If v_∞ is not colored red, then

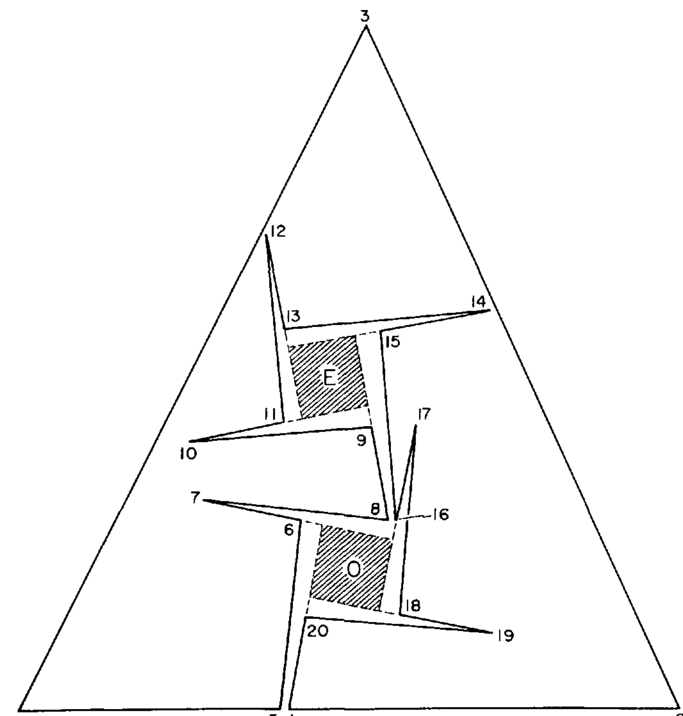

Fig. 6.2. Guards on the even vertices leave E uncovered, and guards on the odd vertices leave O uncovered.

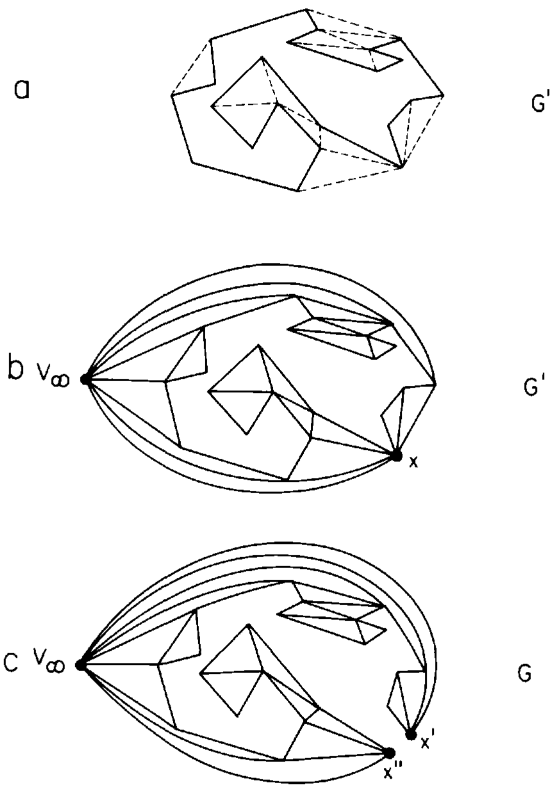

Fig. 6.3. The graph G in (c) is produced from a triangulation of the hull pockets (a) by connecting all hull vertices to v_∞ (b) and splitting some hull vertex x.

placing guards at the red nodes covers the exterior of the original polygon with $\lfloor (n+2)/3 \rfloor \le \lceil n/2 \rceil$ vertex guards.

If, however, v_∞ is colored red, this strategy will not work, since no guard may be placed at v_∞ as it is not a vertex of the polygon. In this case, place guards with the second least frequently used color. Suppose the number of occurrences of the three colors are $a \le b \le c$, with $a + b + c = n + 2$. Since $a \ge 1$, $b + c \le n + 1$. Thus $b \le \lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$.

Finally, in either case, every triangle incident to v_∞ is dominated, and since v_∞ is not guarded, every other hull vertex must be guarded. These guards clearly cover the exterior outside the hull. The exterior inside the hull is covered by the usual 3-coloring argument. □

6.2.2. Orthogonal Polygons

Although more guards are needed to cover the exterior than the interior of an arbitrary polygon ($\lceil n/2 \rceil$ versus $\lfloor n/3 \rfloor$), with orthogonal polygons the numbers required differ only slightly: $\lceil n/4 \rceil + 1$ for the exterior versus $\lfloor n/4 \rfloor$ for the interior.

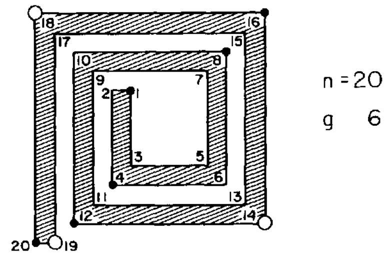

Fig. 6.4. An orthogonal spiral requires $\lceil n/4 \rceil + 1$ vertex guards (solid) to cover the exterior.

THEOREM 6.2 [Aggarwal 1983]. $\lceil n/4 \rceil + 1$ vertex guards are necessary and sufficient to see the exterior of an orthogonal polygon of n vertices.

Proof. Necessity follows from the spiral of $n = 4m$ vertices shown in Fig. 6.4. Starting from the interior of the spiral, it is clear that the vertices labeled 1, 4, 8, 12, 16, . . . , $4(m-3)$ are an optimal choice for guard locations; a second optimal choice is 7, 11, 15, . . . , $4m-1$. At the outside arm there is some choice where to place the last few guards. The first sequence can continue in one of two ways: either . . . , $4(m-2)$, $4(m-1)$, $4m$ (the solid circles in Fig. 6.4), or . . . , $4(m-2)$, $4(m-2)+2$, $4(m-1)+2$, $4m-1$ (the empty circles in the figure). In either case $m+1$ guards are used. Similar reasoning shows that $m+1$ guards are required for other choices of guard locations. If the spiral is extended by two more vertices, $n = 4m+2$, then $m+2$ guards are required. In all cases, $\lceil n/4 \rceil + 1$ guards are required.

The sufficiency proof follows almost directly from the L-shaped partition discussed in Sections 2.5 and 2.6. Given an n vertex orthogonal polygon P, remove the horizontal edge e with largest y-coordinate (or any one with maximum height if there are several), extend the two adjacent vertical edges upward and enclose the entire polygon with a bounding rectangle as illustrated in Fig. 6.5. The interior of this new $n+4$ vertex polygon P'

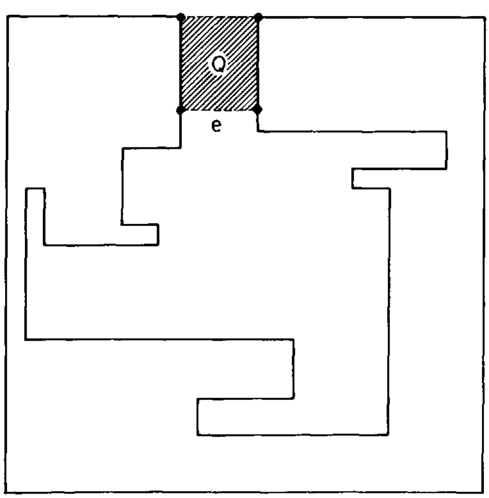

Fig. 6.5. The exterior of an orthogonal polygon may be converted into the interior of another by deleting the highest edge e and enclosing within a rectangle.

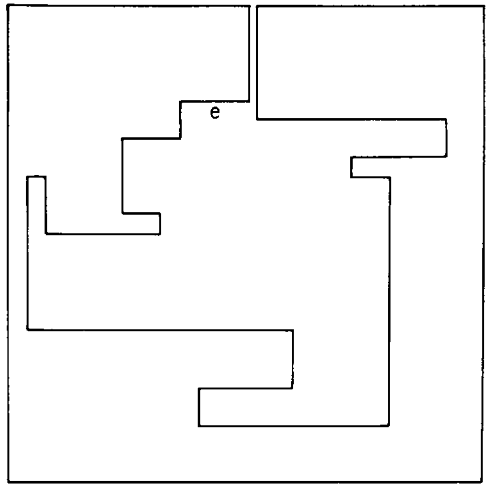

Fig. 6.6. An alternative enclosing strategy, used when n is divisible by 4.

coincides with the immediately surrounding exterior of P, except for the rectangle Q shaded in the figure, which is exterior to both.

The crucial observation is that the guard placement procedure described in Section 2.6 locates guards *only* on reflex vertices. Since the six new vertices of P' are all convex, guards covering the interior of P' will all be located on vertices of P. It should be clear that coverage of the immediate exterior of P by vertex guards implies coverage of the entire exterior: each side of the smallest rectangle enclosing P must have a guard on it, and these guards cover the infinite plane outside the rectangle. By Theorem 2.5, the interior of P' can be covered with $\lfloor (n+4)/4 \rfloor$ such guards. The region Q will need its own guard, yielding a total of $\lfloor n/4 \rfloor + 2$ guards. When $n \equiv 2 (\text{mod } 4)$, this formula is identical to $\lceil n/4 \rceil + 1$. Note that the guard placement must cover the entire infinite plane, as the bounding rectangle can be chosen to be arbitrarily large.

If $n \equiv 0 (\text{mod } 4)$, we have the freedom to augment P' by two vertices without increasing the number of guards, because of the presence of the floor function in Theorem 2.5. Therefore, modify P' to have $n+6$ vertices as shown in Fig. 6.6. Note that now the interior of P' and the immediate exterior of P exactly coincide: Q has been removed. P' can be covered with $\lfloor (n+6)/4 \rfloor = \lfloor n/4 \rfloor + 1$, which coincides with $\lceil n/4 \rceil + 1$ since $n \equiv 0 (\text{mod } 4)$. □

6.2.3. Guards in the Plane

The surprising asymmetry between Theorems 1.1 and 6.1, which respectively state that $\lfloor n/3 \rfloor$ guards suffice to cover the interior of a polygon but $\lceil n/2 \rceil$ may be needed to cover the exterior, can be removed by a loosening of restrictions on the guard placements. We will show in this section that

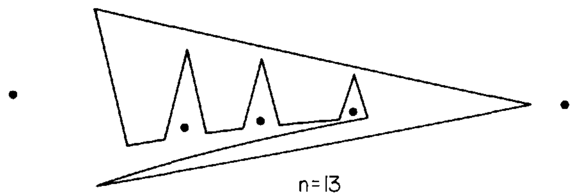

Fig. 6.7. $\lceil n/3 \rceil$ point guards are necessary to cover the exterior of a polygon.

$\lceil n/3 \rceil$ guards are necessary and sufficient to cover the exterior of a polygon if the guards are not restricted to vertices of the polygon, but may be located anywhere in the plane exterior to the polygon or on the polygon boundary. Such guards are called *point guards* to contrast with the vertex guards used in the previous section. This theorem restores a pleasing symmetry between interior and exterior coverage: $\lfloor n/3 \rfloor$ versus $\lceil n/3 \rceil$; even the floor and ceiling operators interchange naturally.

Necessity of $\lceil n/3 \rceil$ guards is established by turning the comb example of Fig. 1.2 "inside-out." Figure 6.7 shows a 13-vertex example that requires $5 = \lceil 13/3 \rceil$ guards. The general construction with k comb prongs has $n = 3k + 4$ vertices are requires $k + 2 = \lceil n/3 \rceil$ guards. Necessity for the other two possible values of $n \bmod 3$ are obtained by adding one or two extraneous vertices to this example.

Sufficiency of $\lceil (n+1)/3 \rceil$ guards is easily established.

LEMMA 6.1. $\lceil (n+1)/3 \rceil$ point guards suffice to cover the exterior of an n vertex polygon P.

Proof. Rotate P so that vertex a is uniquely highest and b uniquely lowest, and add two vertices λ and ρ below the lowest vertex of P, and far enough away to both see a. See Fig. 6.8. Triangulate the interior of the convex hull exterior to the polygon, and add diagonals from λ and ρ to every hull vertex visible to them. Finally split vertex a in two. The result is a triangulated

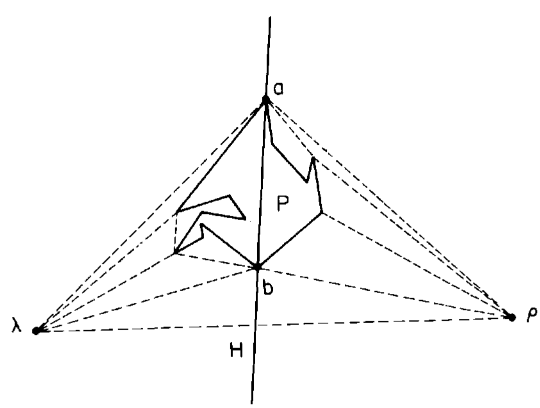

Fig. 6.8. Domination of this graph implies coverage of the exterior of the polygon.

polygon of $n+3$ vertices. Cover this polygon with $\lfloor(n+3)/3\rfloor = \lceil(n+1)/3\rceil$ guards by Theorem 1.1. We now argue that these guards cover the entire exterior.

Clearly the portion interior to the hull is covered. Now consider the left half-plane H determined by a line through a and b. If a guard is assigned to λ, then it covers all of H exterior to the hull. If λ has no guard, then since every hull edge in the left chain between a and b forms a triangle with λ, at least one endpoint of each hull edge must have a guard. But then these guards cover the exterior in H. Applying the same argument to the right half-plane establishes the lemma. □

One of the longest proofs in the first draft of this book was devoted to removing "a third of a guard" from this lemma. The proof followed the same general approach as that used in Lemma 6.1, but difficulties arise because combinatorial dominance of arbitrary exterior triangulations of the type just considered cannot suffice to establish the theorem, as shown in Fig. 6.9. Here $n=6$, so $\lceil n/3\rceil = 2$, but $\lceil(n+1)/3\rceil = 3$. The triangle abc requires one guard in any combinatorial domination. But then two further guards will be required regardless of how the exterior is triangulated. The figure shows that indeed only two point guards are necessary, one at each of λ and ρ, but these guards represent a combinatorial dominance only if the exterior is retriangulated. Our proof involved a long cascade of cases, and an abandonment of combinatorics for geometry at a critical junction (Aggarwal 1984). Fortunately, this theorem has gone the way of Chvátal's theorem and the Kahn, Klawe, Kleitman theorem in that a proof far simpler than the original has been found. Recently Shermer discovered a concise coloring argument, which we present below.

THEOREM 6.3 [Aggarwal and O'Rourke 1984]. $\lceil n/3\rceil$ point guards are sometimes necessary and always sufficient to cover the exterior of a polygon P of $n>3$ vertices.

Proof [Shermer 1986]. Necessity has already been established. If P is convex, then two guards placed sufficiently far away on opposite sides of P suffice to cover the exterior. Since $n>3$, $2 \leq \lceil n/3\rceil$.

Suppose then that P has at least one pocket—that is, an exterior polygon interior to the hull and bound by a hull edge. Rotate P so that a and b are uniquely the highest and lowest vertices, respectively, and add two new

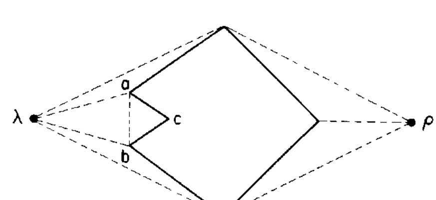

Fig. 6.9. A graph that requires 3 combinatorial guards for dominance, but only 2 point guards are needed for coverage of the exterior.

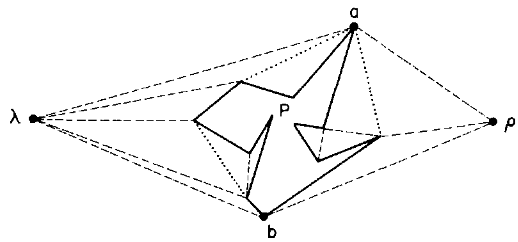

Fig. 6.10. If the number of hull vertices is even, this graph is 3-colorable.

vertices λ and ρ on opposite sides of P and sufficiently distant so that they both can see both a and b. Triangulate the pockets of the hull, and connect λ and ρ to all visible hull vertices, as shown in Fig. 6.10. The strategy is as follows. Since $\lceil n/3 \rceil = \lfloor (n+2)/3 \rfloor$, if we could show that the constructed triangulation graph G were 3-colorable, placing guards with the least frequently used color would establish the theorem via the same argument for exterior coverage used in Lemma 6.1. But G is not always 3-colorable: the graph in Fig. 6.9 requires four colors. Shermer's idea is to modify the graph so that it is always 3-colorable.

Let h be the number of hull vertices. Two cases will be considered: h even and h odd.

Case (h even). G is always 3-colorable in this case, regardless of the choice of λ and ρ. To see this, color the hull vertices alternately with colors 1 and 2. Then λ and ρ may be colored 3. Each pocket lid (shown dotted in Fig. 6.10) is an edge of the polygon forming the pocket. With the constraint that the lid is colored $(1, 2)$, a pocket polygon may be 3-colored as in Section 1.3.1. The result is a 3-coloring of G, implying domination with $\lfloor (n+2)/3 \rfloor$ guards. By the argument in the proof of Lemma 6.1, these guards cover the entire exterior.

Case (h odd). The above approach will not work, as the hull cannot be 2-colored. Let yz be a pocket lid, and let x be the apex of the triangle in the pocket supported by yx. Now orient P and choose ρ so that

(1) ρ can see x,
(2) $x\rho$ is not parallel to any polygon edge, and
(3) ρ is distant enough to see an "antipodal" pair of vertices a and b, vertices that admit parallel lines of support.

Condition (2) is imposed so that a and b are ensured to become uniquely highest and lowest as before. Clearly it is always possible to choose such a ρ. Place λ on the opposite side of P as in the previous case. See Fig. 6.11. The quadrilateral ρyxz is convex by construction. Delete diagonal yz from G, and add diagonal ρx. (This is the necessary retriangulation mentioned earlier.) If we now consider x part of the hull, we have increased the number of hull vertices to be even. Now proceed as in the previous case. Color the cycle of hull vertices and x colors 1 and 2 in alternation, color λ

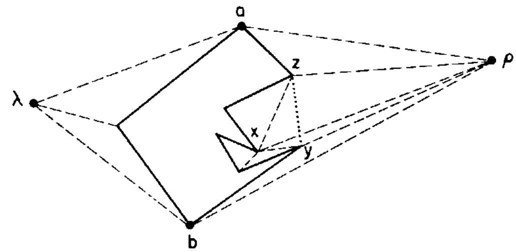

Fig. 6.11. If the number of hull vertices is odd, one vertex x within the hull is connected to ρ to make a 3-colorable graph.

and ρ color 3, and 3-color all pocket polygons. Dominate G with $\lfloor (n+2)/3 \rfloor$ guards. The argument for exterior coverage used in the proof of Lemma 6.1 must be modified slightly, since it may be that neither endpoint of the hull edge yz is assigned a guard. But in that case, x will be assigned a guard, and the complete exterior is still covered. □

Note that the proof uses at most two guards in the plane strictly exterior to the polygon; the remainder are located at vertices.

6.3. PRISON YARD PROBLEM

How many vertex guards are needed to simultaneously see the exterior and interior of a polygon P of n vertices? An interior point x is seen by a guard at vertex z if the segment zx does not intersect the exterior of P, and an exterior point y is seen by z if zy does not intersect the interior of P. It should be emphasized that the guards are restricted to vertices; the problem is quite a bit different (and easier) if the guards are permitted to be located arbitrarily far away from the polygon. Permitting the guards to be anywhere on the boundary of the polygon does not seem to affect the problem's complexity, so we will only consider vertex guards.

6.3.1. General Polygons

The worst case known is the same as that for the fortress problem: a convex n-gon (Fig. 6.1). This establishes that $\lceil n/2 \rceil$ guards are occasionally necessary. The only sufficiency result so far obtained is rather weak and inelegant:

THEOREM 6.4 [O'Rourke 1983]. For a multiply-connected polygon of n vertices, r of which are reflex, and h of which are on the convex hull of the polygon,

$$\min(\lceil n/2 \rceil + r,\ \lfloor (n + \lceil h/2 \rceil)/2 \rfloor,\ \lfloor 2n/3 \rfloor)$$

vertex guards are sufficient to see both the interior and exterior. Note that,

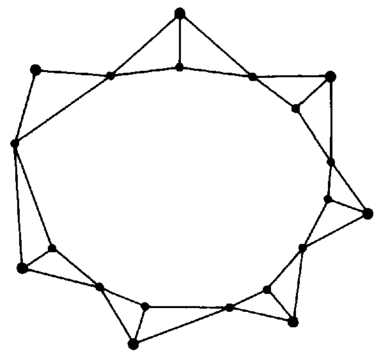

Fig. 6.12. Addition of $\lceil h/2 \rceil$ nodes exterior to the hull leads to $n/2+h/4$ guards (modulo floors and ceilings).

ignoring ceilings and floors, the above formula may be written in the more revealing form

$$n/2 + \min(r, h/4, n/6)$$

Proof. Each of the three formulas is derived by a separate method.

(1) $\lceil n/2 \rceil + r$

Use Theorem 6.1 to cover the exterior with $\lceil n/2 \rceil$ guards, and Theorem 1.5 to cover the interior when $r \geq 1$. If $r = 0$, the polygon is convex and $\lceil n/2 \rceil$ clearly suffice.

(2) $\lfloor (n + \lceil h/2 \rceil)/2 \rfloor$

Triangulate the interior of the convex hull, including both the interior of the polygon and the exterior within the hull. Add $\lceil h/2 \rceil$ new vertices outside the hull, each (except for perhaps one) adjacent to three hull vertices, as shown in Fig. 6.12. The resulting graph is planar and so may be 4-colored. Place guards at nodes colored with the two least frequently used colors. Together these colors cannot be used more than half of the total number of vertices—that is, not more than $\lfloor (n + \lceil h/2 \rceil)/2 \rfloor$. Since every triangle has three differently colored vertices, at least one guard is in the corner of each triangle. If any of the outside-hull vertices are assigned guards, place the guard on the middle of its three adjacent hull vertices, or either one if only adjacent to two. The inside of the hull is covered because it is partitioned into triangles; the exterior of the hull is covered because every hull edge has a guard on at least one endpoint.

(3) $\lfloor 2n/3 \rfloor$

Triangulate the interior of the hull as above. Add a single new vertex v_∞ outside the hull, and connect it to every hull vertex (Fig. 6.13). The resulting graph is planar and can be 4-colored. Place guards on the two least frequently used colors not matching the color assigned to v_∞. Suppose the

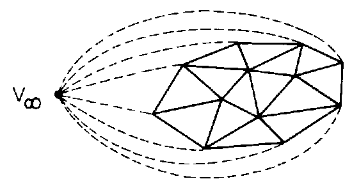

Fig. 6.13. Connecting every hull vertex to v_∞ leads to $\lfloor 2n/3 \rfloor$ guards.

number of nodes colored i is c_i, so that

$$c_1 \leq c_2 \leq c_3 \leq c_4, \tag{1}$$

$$c_1 + c_2 + c_3 + c_4 = n + 1. \tag{2}$$

If v_∞ is colored 1, 2, 3, or 4, the two colors used for guards occur $(c_2 + c_3)$, $(c_1 + c_3)$, $(c_1 + c_2)$, and $(c_1 + c_2)$ times, respectively. Clearly the first case is the worst because of the inequality (1). In this case, since $c_1 \geq 1$, Equation (2) becomes

$$c_2 + c_3 + c_4 \leq n$$

and Equation (1) implies that $c_2 + c_3 \leq \lfloor 2n/3 \rfloor$.

Now the argument is the same as above: every triangle has at least one guard at a vertex, and the exterior triangles are likewise covered without using a guard at v_∞. □

If the guards are not restricted to lie on the boundary of the polygon, then it is easy to establish $\lfloor n/2 \rfloor + 1$ sufficiency: triangulate the interior of the hull, and add two vertices sufficiently far outside the hull to connect every hull vertex by a straight line (as in Fig. 6.10, but with interior triangulation as well); 4-color the resulting graph and place guards on the two least frequently used colors.

It seems, however, that this freedom to place guards outside of the polygon is not needed:

CONJECTURE 6.1. $\lceil n/2 \rceil$ vertex guards are sufficient to see the interior and exterior of a polygon of n vertices.

Proving or disproving this conjecture is one of the most interesting open problems in this field.

Before proceeding to orthogonal polygons, it will be instructive to prove $\lfloor 2n/3 \rfloor$ sufficiency by a different method. Cover the exterior with $\lceil n/2 \rceil$ guards according to Theorem 6.1. Now triangulate the interior of the polygon, and remove all vertices assigned a guard and their incident edges. The resulting graph may be disconnected (as in Fig. 6.14), but it will have a total of no more than $n - \lceil n/2 \rceil = \lfloor n/2 \rfloor$ nodes. Further delete any vertices or edges of this graph that are not part of any triangle; these clearly are covered from all sides by guards. Now apply the interior visibility argument: 3-color and place guards at the least frequently used color. Note that although the components are not necessarily triangulation graphs of simple

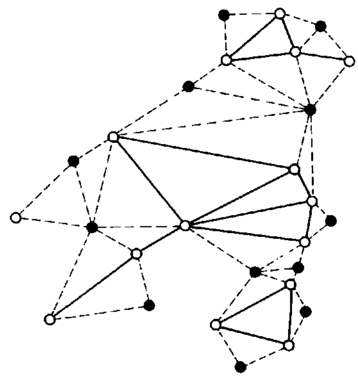

Fig. 6.14. The dashed edges are incident on a guarded vertex (solid dots).

polygons, they are nevertheless 3-colorable, as each is a subgraph of such a graph. This requires at most an additional $\lfloor n_i/3 \rfloor$ guards for each connected component of n_i nodes; note that $n_i \geq 3$. Thus the total number of guards required is $\lceil n/2 \rceil + \lfloor \lfloor n/2 \rfloor /3 \rfloor$. This formula equals $\lfloor 2n/3 \rfloor$, except when $n \equiv 1 (\mathrm{mod}\ 6)$, in which case it equals $\lfloor 2n/3 \rfloor + 1$. A similar argument will be used in the next section.

6.3.2. Orthogonal Polygons

Clearly the separate interior and exterior visibility results for orthogonal polygons (Theorems 2.2 and 6.2) may be combined to yield $\lfloor n/4 \rfloor + \lceil n/4 \rceil + 1 \leq \lceil n/2 \rceil + 1$ sufficiency for vertex guards. This straightforward combination of interior and exterior results may be improved slightly by following the above sketch of the alternate $\lfloor 2n/3 \rfloor$ proof. Interestingly, the proof employs both methods known for achieving the interior orthogonal result.

THEOREM 6.5 [O'Rourke 1983]. $\lfloor 7n/16 \rfloor + 5$ vertex guards are sufficient to see both the interior and exterior of a simple orthogonal polygon.

Proof. Cover the exterior with $\lceil n/4 \rceil + 1$ guards according to Theorem 6.2. Now partition the interior into convex quadrilaterals as guaranteed by Theorem 2.1. Discard every edge that has a guard at one of its endpoints. This may disconnect the graph, but the total number of vertices is no more than $n - (\lceil n/4 \rceil + 1)$; see Fig. 6.15. Further delete all vertices and edges that are not part of any quadrilateral; this is justified since guards cover all sides of such edges and around all such vertices. Four-color the pieces (as was done in the proof of Theorem 2.2) and place guards on the least frequently used color. Each piece of n_i nodes will need no more than $\lfloor n_i/4 \rfloor$ guards; note that $n_i \geq 4$. The total number of guards used is

$$\lceil n/4 \rceil + 1 + \left\lfloor \frac{n - \lceil n/4 \rceil - 1}{4} \right\rfloor .$$

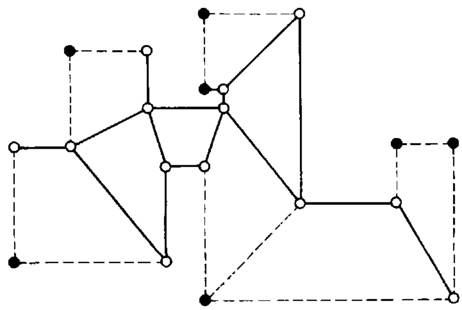

Fig. 6.15. The dashed edges are adjacent to a guard as in Fig. 6.14.

This expression is no larger than $\lfloor 7n/16 \rfloor + 5$; the constant 5 can be reduced for certain values of n mod 16. □

This result appears to be nearly as weak as the $\lfloor 2n/3 \rfloor$ result for arbitrary polygons.

6.4. ALGORITHMS

Because the partitioning theorems in this chapter employ techniques from previous chapters, such as triangulation, 3-coloring, and L-shape partitioning, few new algorithmic questions are raised. The only new technique used is 4-coloring of a planar graph. Unfortunately, Appel and Haken's proof leads to a rather complex $O(n^2)$ algorithm (Appel and Haken 1977; Frederickson 1984). Using this algorithm with algorithms discussed in previous chapters, we arrive at the worst-case time complexities shown in Table 6.1. Here $O(T)$ is the time complexity for triangulation; the current best result is $T = n \log \log n$.

Table 6.1

Problem	Techniques	Guards	Time
Fortress			
general	triangulation, 3-coloring	$\lceil n/2 \rceil$	$O(T)$
orthogonal	L-shaped partition	$\lceil n/4 \rceil + 1$	$O(T)$
Prison Yard			
general	exterior	$\lceil n/2 \rceil + r$	$O(T)$
	triangulation, 4-coloring	$\lfloor (n + \lceil h/2 \rceil)/2 \rfloor$	$O(n^2)$
	triangulation, 4-coloring	$\lfloor 2n/3 \rfloor$	$O(n^2)$
	exterior, triang., 3-coloring	$\lfloor 2n/3 \rfloor + 1$	$O(T)$
orthogonal	exterior, quadrilateralization, 4-coloring	$\lfloor 7n/16 \rfloor + 5$	$O(T)$

Although 4-coloring is a time consuming process, several linear algorithms are known for 5-coloring (Chiba *et al.* 1981; Frederickson 1984). Using one of these algorithms improves the speed from $O(n^2)$ to $O(T)$ in two instances in the table, but increased the number of guards: since guards must be placed with three out of the five colors to guarantee that every triangle receives a guard, the number of guards becomes $\lfloor 3(n + \lceil h/2 \rceil)/5 \rfloor$ and $\lfloor 3n/4 \rfloor$, respectively.

6.5. NEGATIVE RESULTS

Two natural approaches to solving the prison yard problem lead to conjectures which, if true, would either settle the problem or represent a significant advance. This section formulates these conjectures and presents counterexamples.

6.5.1. Triangulation

Triangulate the interior of an n-vertex polygon (including the interior of any holes) and the exterior inside the hull. Now extend rays to infinity from each hull vertex.

*Conjecture. Each triangle and each unbounded region of the above described figure can be dominated by $\lceil n/2 \rceil$ combinatorial vertex guards.

If this conjecture were true, then the prison yard problem would be solved for polygons with holes. Figure 6.16 is, however, a counterexample. There is just one Hamiltonian cycle of the triangulation: the convex hull. Thus this figure can only arise from a convex decagon, which can be easily covered

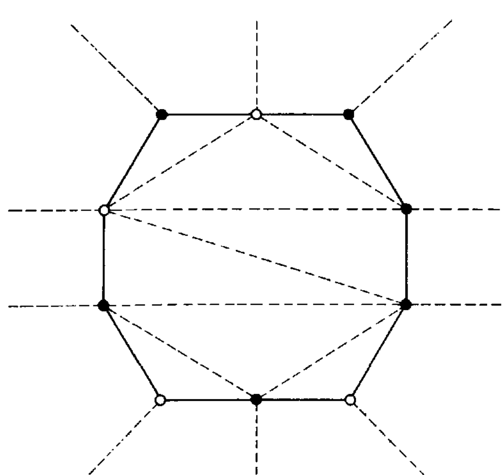

Fig. 6.16. A 10-node graph that requires 6 combinatorial guards (solid dots) for domination.

with five guards. But since there exists a triangulation of this decagon (as shown) that cannot be dominated by five guards, one cannot hope to prove $\lceil n/2 \rceil$ sufficiency starting with an arbitrary triangulation.

6.5.2. Convex Partitioning

Another method of reducing the problem to combinatorics is to form a convex partitioning of the plane from the n-vertex polygon. Such partitions of the interior of the polygon were considered by Chazelle (Chazelle 1980) and discussed in Section 1.4. At each vertex of the polygon, extend a ray that bisects the angle at the vertex outward if the vertex is convex and inward if it is reflex. The ray is extended to its first intersection with a polygon edge on another ray. The bisection procedure may be applied to the vertices in any order. Chazelle proved (for interior partitionings) that the resulting graph can be made cubic (each node has degree 3) by slightly varying the ray angles to avoid "coincidences." The same result holds for the interior and exterior partitioning. An example is shown in Fig. 6.17.

LEMMA 6.5. Each n-vertex polygon (with or without holes) induces a cubic convex partitioning of the plane into $n + 1$ regions.

Proof. That the partition can be chosen so that its graph is cubic follows from Chazelle's method (Chazelle 1980). That the number of regions is $n + 1$ is established by the following argument. Initially there are $h + 2$ regions: the exterior of the polygon, its interior, and the interior of each of the h holes. The first ray towards the exterior of the polygon does not increase the number of regions; similarly the first ray from each of the h holes towards the exterior of the holes cannot increase the number of

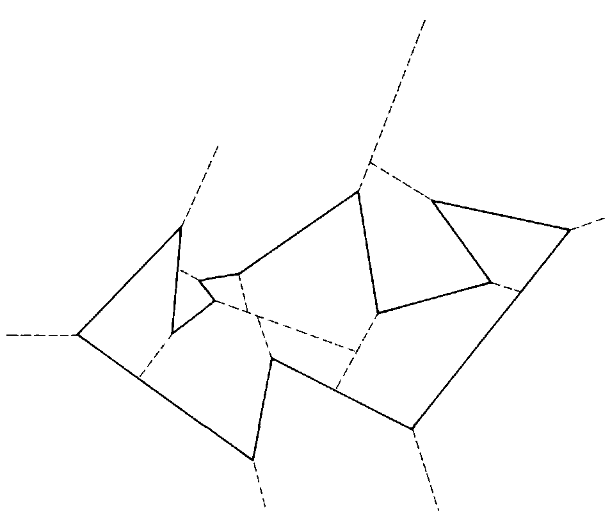

Fig. 6.17. A convex partitioning of the plane induced by a polygon; here $n = 14$ and 15 regions result.

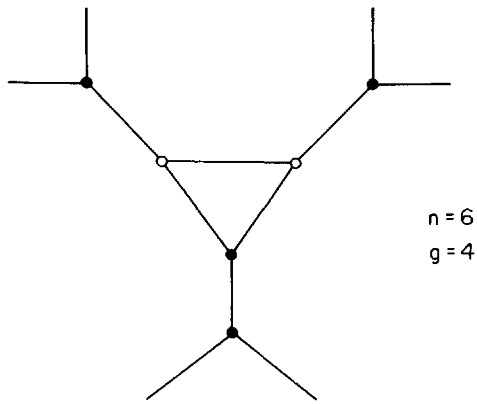

Fig. 6.18. A convex partitioning of 6 vertices that requires 4 combinatorial guards (solid dots) for domination.

regions. Each of these $h+1$ cuts can be thought of as reducing the multiply-connectedness by 1 without increasing the number of regions. Each of the remaining $n-(h+1)$ cuts increments the number of regions by 1. So the total number of regions is $(h+2)+[n-(h+1)]=n+1$. □

Now each vertex of the original polygon sits at the junction of precisely three convex regions. A guard placed at a vertex covers all three regions entirely, since they are convex. So the task is to select $\lceil n/2 \rceil$ vertices, each covering three regions, such that all $n+1$ convex regions are covered by at least one vertex. This scheme leads to the following conjecture.

*Conjecture. Any convex partitioning of the plane into $n' = n+1$ regions can be dominated with $\lfloor n'/2 \rfloor = \lceil n/2 \rceil$ vertex guards.

This conjecture is both stronger and weaker than what is needed to prove $\lceil n/2 \rceil$ sufficiency for the prison yard problem. It is stronger in that its claim is for *all* convex partitions, but those arising from the angle bisection technique represent only a subclass (for instance, an n-vertex polygon will generate a partition with no more than $2n$ edges). It is weaker in that only the polygon vertices are candidates for guard location: ray-ray intersections may lay arbitrarily far outside of the polygon. Nevertheless, the conjecture would represent an advance if true, and would be interesting in its own right. Figure 6.18 shows a simple counterexample, however. There are $n=6$ vertices and $2n=12$ edges, but $g=4>\lceil n/2 \rceil =3$ guards are necessary. This figure was derived from its non-Hamiltonian dual graph, which can be seen to have no perfect matching after three regions are covered by one guard, by Tutte's theorem (Tutte 1947; Harary 1969).[1] The partitioning in the

1. A *matching* is a collection of edges that share no nodes. A *perfect matching* is one that matches every node. Tutte's theorem states that a graph G has a perfect matching iff it has an even number of nodes and there is no set S of nodes such that the number of odd components of $G-S$ exceeds $|S|$. If there is an S such that the number of components (even or odd) of $G-S$ exceeds $|S|$, then G is non-Hamiltonian.

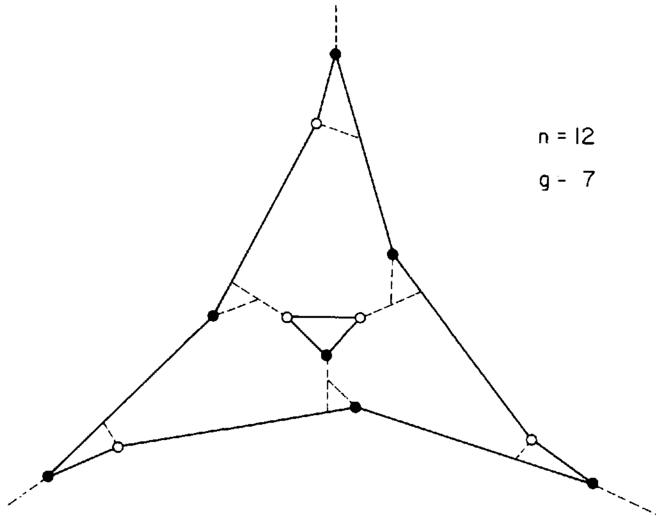

Fig. 6.19. A convex partitioning of 12 vertices that requires 7 combinatorial guards (solid dots) for domination.

figure could never arise from the bisection process, however, as that process always produces regions with at least three bounding edges each. This suggests tightening the conjecture to convex partitions whose dual graphs have degree at least 3. Again, however, a counterexample can be derived from a non-Hamiltonian dual that has no perfect matching after deletion of three nodes with one guard, as shown in Fig. 6.19. This figure has $n = 12$ vertices, $n' = n + 1 = 13$ regions, and $21 < 2n$ total edges. It requires $g = 7 > \lceil n/2 \rceil = 6$ guards. In addition, this partitioning *can* arise from the bisection process: from a polygon with one hole, as shown in the figure.

This last figure definitely establishes that this attempted reduction to combinatorics cannot prove $\lceil n/2 \rceil$ sufficiency for multiply-connected polygons; it remains possible that the conjecture holds for convex partitions arising from polygons with no holes.

6.5.3. Monotone Polygons

We will end this pessimistic section on a positive note by showing that convex partitions lead to a proof of the prison yard conjecture for the special case of monotone polygons.

THEOREM 6.6 [O'Rourke 1984]. $\lceil n/2 \rceil$ guards are occasionally necessary and always sufficient to see the interior and exterior of a monotone polygon of n vertices.

Proof. Assume that the polygon is monotone with respect to the y axis. It will be no loss of generality to assume that the vertices extreme in the y direction are unique; call the highest a and the lowest b. Partition the plane

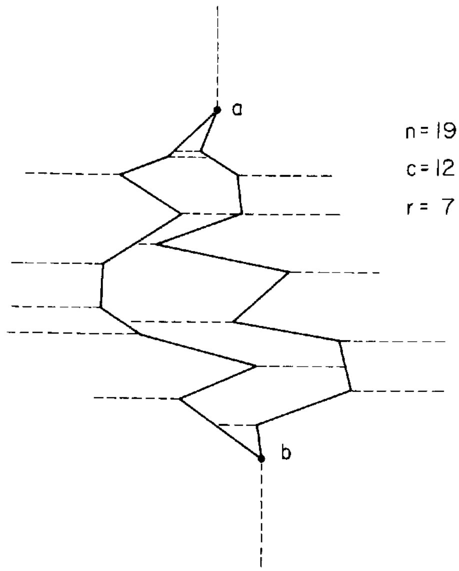

Fig. 6.20. A convex partitioning of the plane induced by a monotone polygon with 12 convex and 7 reflex vertices.

into convex regions as follows. Draw vertical rays from b and a towards the exterior of the polygon, and draw horizontal rays from each convex vertex, again towards the exterior. Finally, draw horizontal segments from each reflex vertex towards the interior of the polygon to the polygon's opposite chain. See Fig. 6.20.

Note that the exterior regions form a ring around the polygon, with each pair of adjacent exterior regions sharing a convex vertex. Thus, if there are c convex vertices, the exterior regions can be covered with $\lceil c/2 \rceil$ guards, one on every other convex vertex. We will use the convention that a is assigned a guard; clearly we have that flexibility. The interior regions form a vertical stack, with each pair of adjacent regions sharing a reflex vertex. If there are r reflex vertices, then there are at most $r+1$ interior regions; this maximum is achieved when no two reflex vertices have the same y-coordinate. Because a guard is placed at a, the top interior region is already covered, leaving r to be covered. These can be covered with $\lceil r/2 \rceil$ guards, one on every other reflex vertex (sorted by their y-coordinate, not their position in the polygon's boundary).

Now $n = c + r$, and we have just established that all regions can be covered with $\lceil c/2 \rceil + \lceil r/2 \rceil$ guards. If at least one of c or r is even, then this formula accords with $\lceil n/2 \rceil$, and we are finished. If both are odd, however, the two ceilings yield one more guard than $\lceil n/2 \rceil$. But, since c is odd, b must be assigned a guard either clockwise from a or counterclockwise from a, choosing every other one. Therefore, we can assume that b is assigned a guard when c is odd. Now the bottom interior region is covered, and only

$r-1$ contiguous interior regions need to be covered. Since $\lceil c/2 \rceil + \lceil (r-1)/2 \rceil = \lceil n/2 \rceil$ when both c and r are odd, we have established the lemma. □

Several miscellaneous problems involving exterior visibility will be considered in Chapter 10.