A 2-CHAIN CAN INTERLOCK WITH AN OPEN 11-CHAIN

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Abstract. An open problem posed in [3] asked for the minimal k such that a flexible open k-chain can interlock with a flexible 2-chain. It was conjectured in [3] that the minimal k satisfies $6 \leq k \leq 11$. In a previous preprint [5], we establish that k exists, and in particular, that $k \leq 16$. Here we improve this result to $k \leq 11$ by proving that a flexible 2-chain can interlock with a flexible, open 11-chain. We offer some reasons to believe that $k = 11$ is minimal.

1. Introduction

A *polygonal chain* (or *chain*) is a linkage of rigid bars (line segments, edges) connected at their vertices (joints, endpoints), which forms a simple path (an *open chain*) or a simple cycle (a *closed chain*). A *folding* of a chain is a reconfiguration obtained by moving the vertices so that the lengths of edges are unchanged and the edges do not intersect or pass through one another. When the vertices act as universal joints, these are *flexible chains*. A collection of chains are said to be *interlocked* if they cannot be separated by foldings.

The interlocking of polygonal chains was studied in [4, 3], establishing a number of results regarding which collections of chains can and cannot interlock. Concerning pairs of open chains, the one hole in the results was posed as an open problem in [3]: what is the minimal number k such that an open, flexible k-chain can interlock with a flexible 2-chain? The assumption behind this question—that there is some k that permits interlocking—was established in [5]: there is an open 16-chain that achieves interlocking. It was conjectured in [3] that the minimal k satisfies $6 \leq k \leq 11$. Here we prove that there is an 11-chain that does interlock with a 2-chain, thus justifying the upper end of that conjectured range. We suspect that $k = 11$ is, in fact, minimal, a point to which we return in the final section. Our proof will follow the overall plan of [5], but requires a more delicate argument.

A result crucial to the construction of our open 11-chain from [3] is:

> A flexible open 3-chain can interlock with a flexible open 4-chain.

The construction, which we call a *3/4-tangle*, is repeated in Figure 1.

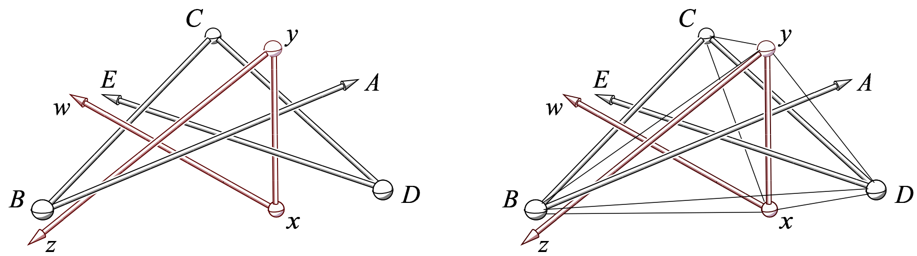

FIGURE 1. Fig. 6 from [3].

We need some facts about the 3/4-tangle:

(1) Proved in Theorem 11 of [3]: the convex hull $CH(B,C,D,x,y)$ of the joints B, C, D, x, and y does not change its combinatorial structure.
(2) Corollary 1 of [5]: For $\epsilon > 0$, choose the lengths of the middle bars $BC = CD = xy = \frac{1}{6}\epsilon$, and the lengths of the end bars $AB = DE = xw = yz = \frac{1}{2}\epsilon$. Let p be the midpoint of xy. Then all joints B, C, D, x, y and endpoints A, E, w, z stay inside the ϵ-ball centered at p.

It is this 3/4-tangle we use in the construction of our open 11-chain, which establishes our main theorem, that a 2-chain can interlock an open 11-chain (Theorem 1 below.)

2. IDEA OF PROOF

The main idea of the proof is to build a "rigid" triangular frame with small rings at its vertices (T_1, T_2, T_3). Such a structure could interlock with a 2-chain, as illustrated in Figure 2(a). For then pulling vertex v of the 2-chain away from the triangle would necessarily diminish the half apex angle α, and pushing v down toward the triangle would increase α. But the only slack provided for α is that determined by the diameter of the rings. We make as our subgoal, then, building such a triangle.

We can construct a triangle with three links, as in Figure 2(b). At the apex, we want to take one subchain aa' and pin its crossing with another subchain bb' to some small region of space. See Figure 2(c) for the idea. It is this pinning that can be achieved by the 3/4-tangle. So the idea is replace the vertex T_1 with a small copy of this configuration. This can be accomplished with 7 links for a 3/4-tangle, but sharing with one of the incident incoming and outgoing triangle links potentially reduces the number of links needed. We can achieve 6 links at the

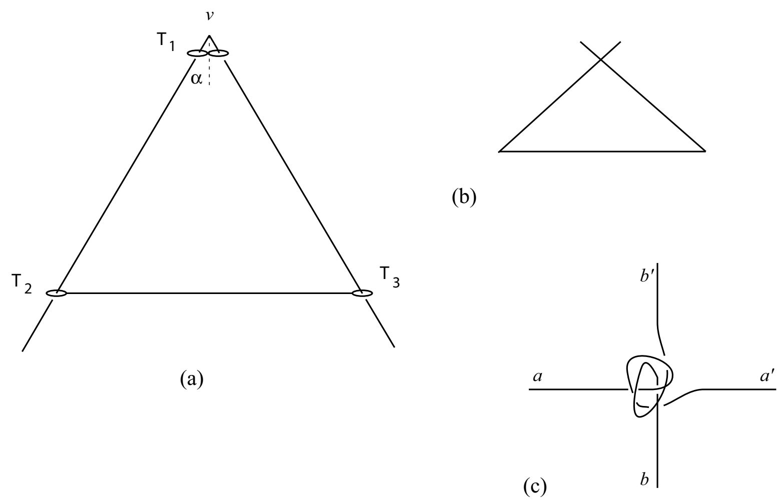

FIGURE 2. (a) A rigid triangle with rings could interlock with a 2-chain; (b) An open chain that simulates a rigid triangle; (c) Fixing a crossing of aa' with bb'.

tangle near T_1. The other two vertices of the triangle need to simulate the rings T_2 and T_3 in Figure 2(a), and this can be accomplished with one extra link per vertex. Together with the 3 links for the main triangle skeleton, we employ a total of $3 + (6 + 1 + 1) = 11$ links.

3. A 2-CHAIN CAN INTERLOCK WITH AN 11-CHAIN

We take a 3/4-tangle whose joints and end points of the pair all stay within an ϵ-ball centered at the midpoint of the middle link of the 3-chain as in Fact (2). Position the tangle as the summit vertex of a triangle with the links arranged as shown in Figure 3.

The configuration in Figure 3 is realizable,[1] that is, there is more than enough flexibility in the design to ensure that va and vb can indeed share the same 2-chain apex v. To see that this is so, notice that the 3-chain and the 4-chain in the 3/4-tangle (cf. Figure 1) can be made to lie in planes that are nearly orthogonal. If we arrange the 3-chain to be almost parallel to the plane of the 2-chain, and twist the 4-chain to be almost orthogonal to that plane, then we can thread the 2-chain as in Figure 3 to pass through the two jag loops that make up the 4-chain, at the same time as weaving through the two jag loops at the base of the triangular frame from above, as depicted.

[1] And indeed we have constructed physical models of it.

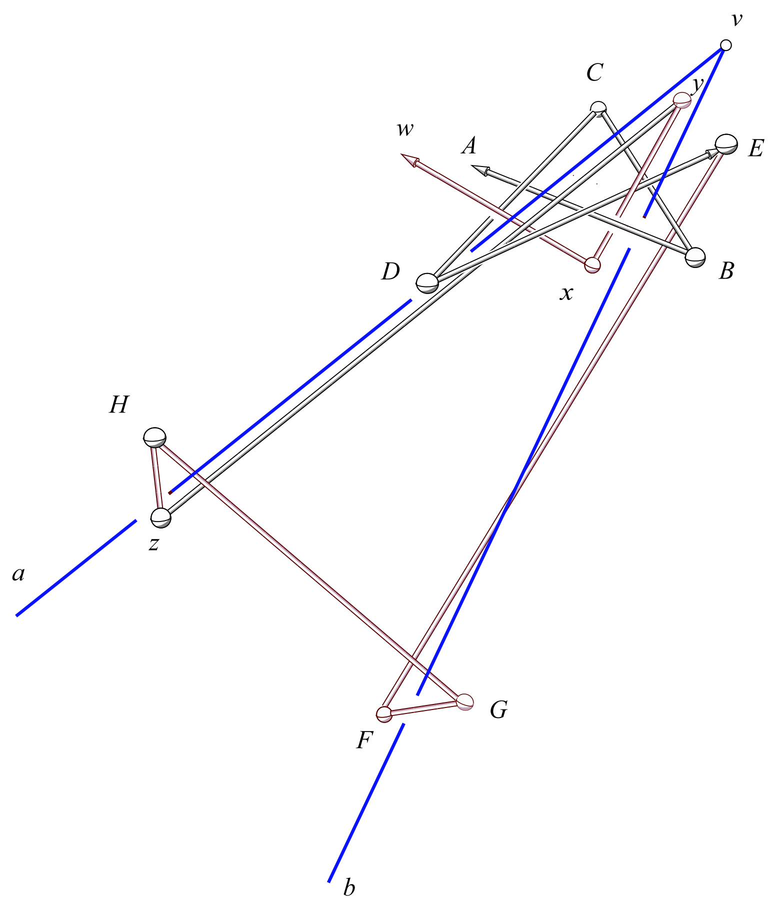

FIGURE 3. An open 11-chain forming a nearly rigid triangle interlocking a 2-chain.

3.1. **2-chain Through Triangle Jag Corners.** In fact, the link va of the 2-chain passes through an ϵ-ball centered at z, and the link vb passes through an ϵ-ball centered at F. We call the simple structure at the two base corners a *jag loop*, and focus on the jag loop including the short link Hz. Consider the jag loop as a triangle with vertices H, z and the "intersection" of HG and zy. The jag loop could only become large if zy and HG were nearly parallel, which cannot happen because of the near-rigidity of the overall triangular structure. Near-rigidity follows because the long links EF, HG and zy have a fixed length, and the connecting links are short, and so the overall triangular structure cannot "collapse." Hence we can ensure that the loop is confined to an ϵ-ball about z by making $|Hz| < \epsilon$. In addition, it was established in Corollary 2 of [5] that va interlocks with the jag under the assumption that va is nearly parallel to yz. We conclude that the link va passes through an ϵ-ball at z. Similarly, the link vb passes through an ϵ-ball centered at F.

3.2. **The 2-chain Links Are Trapped by the 3/4-tangle.** Thus the only way the 2-chain could slide free of the triangular frame is if one of the end vertices enters the ϵ-ball at the jag loop corners.

Lemma 1. *The vertex v of the 2-chain cannot "unweave" from the 3/4-tangle. Thus the links of the 2-chain are trapped by the 3/4-tangle.*

Proof. Links vb and va pierce $\triangle BCD$ and straddle BA, and (a, v, b) surrounds DE. Thus (a, v, b) and (x, y, z) weave through the tangles in exactly the same manner, a fact we will exploit below. Now consider the 3/4-tangle $(A, B, C, D) \cup (x, y, z)$ in isolation, as if it were not part of a larger structure. We know from Fact (1) above that $H = CH(B, C, D, x, y)$ cannot change, and furthermore, Lemma 8 of [3] establishes (as detailed in the proof of their Theorem 11) that y cannot penetrate the tetrahedron $T = CH(A, B, C, D)$ unless H changes. Thus, y cannot penetrate the tetrahedron T. See Figure 4. We now obtain a contradiction by assuming that v can unweave from

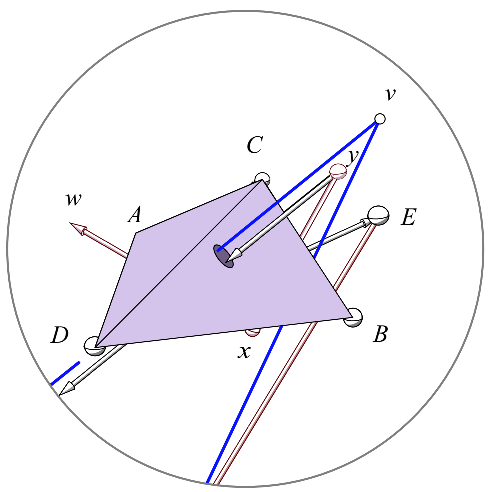

FIGURE 4. If v can move to penetrate the tetrahedron T, so can y.

the tangle.

The only way v can unweave is for v to penetrate T, for otherwise it remains looped around AB (which is confined to the tangle by Fact (2) above). First note that, because of the similar weaving patterns of (a, v, b) and (x, y, z), we could move (a, v, b) to be as nearly coincident to (x, y, z) as desired: v could be moved close to y, and av and bv to near parallelism with zy and xy respectively. Let this motion, from the initial position, be m_1. Now, assume for the contradiction, that there is some motion m_2 from the initial position for v to penetrate T, a motion of the full 2-chain embedded within the full 11-chain. Consider the effect of the motion $m_1^{-1} \circ m_1$ applied to (x, y, z) just as part of the 3/4-tangle. It first moves y to v's initial position, and then tracks the motion of v that penetrates T. The reason this motion can apply to (x, y, z) is that (a, v, b) is more constrained than (x, y, z), for the former is entangled in the full 11-chain, while the latter, we are assuming, is in the 3/4-tangle, a subpart of the 11-chain. Thus, the purported motion of v would serve as a motion for y to penetrate T, which we know is impossible. □

3.3. **Apex v Cannot Move Far.** Thus the 2-chain (a, v, b) cannot slide free unless v is pulled out so that one of it end vertices a or b enters the ϵ-ball containing the triangle base corner. We argue below that this cannot occur. We recall Lemma 4 from [5]; see Figure 5(a):

When ϵ is sufficiently small, a line piercing two disks of radius ϵ can angularly deviate from the line connecting the disk centers at most $\delta \leq \frac{2\epsilon}{x}$, where x is the distance between the disk centers. Figure 5 (a) illustrates the largest angle δ.

Let O, A, B be the centers of the three ϵ-balls containing the triangle corners; see Figure 5(b). Let the base $|AB| = 2L$, the summit angle $\angle AOB = 2\theta$, the base angles $\angle OAB = \angle OBA = \beta$, and the altitude $|OZ| = h$. The following lemma captures the key constraint on motion of the 2-link.

Lemma 2. *If the links of the 2-chain vab intersect the ϵ-disks as illustrated, then the distance $d(v, AB)$ from v to AB approaches h as $\epsilon \to 0$.*

Proof. Figure 5(c) illustrates the greatest distance $d(v, AB) = h_{max}$ and 5(d) illustrates the least distance $d(v, AB) = h_{min}$. From Figure 5(d) we see that h_{min} occurs with $\triangle wvZ$ having apex angle of $\theta + \delta$ and base angle of $\beta - \delta$. Let $2y$ be the amount by which the base $2L$ is lengthened. Using $\triangle wAa$ we find that $\sin(\beta - \delta) = \frac{\epsilon}{y}$. Using $\triangle wvZ$ we find that $\tan(\beta - \delta) = \frac{h_{min}}{L+y}$. Thus we have

$$\begin{aligned} h_{min} &= \tan(\beta - \delta)(L + y) = \tan(\beta - \delta)\left(L + \frac{\epsilon}{\sin(\beta - \delta)}\right) \\ &= L \tan(\beta - \delta) + \frac{\epsilon}{\cos(\beta - \delta)}. \end{aligned}$$

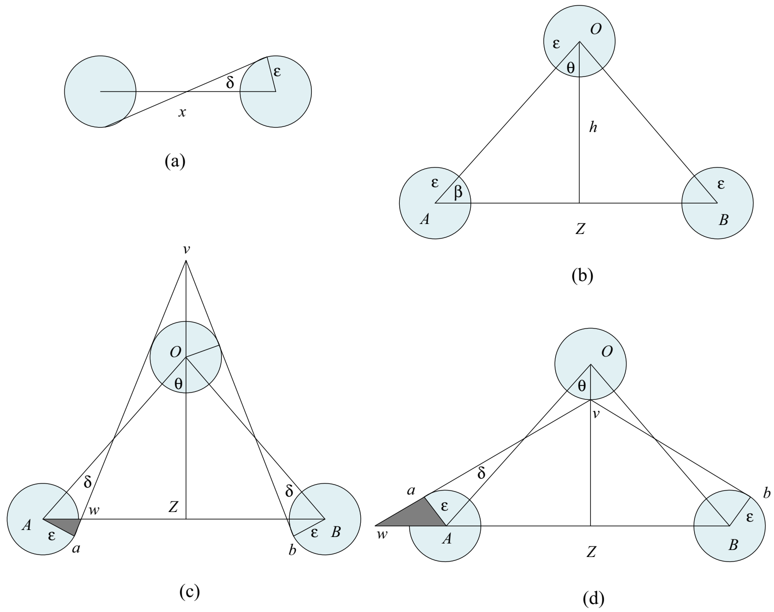

FIGURE 5. Triangle Lemma: (a) Line through two disks deviates at most δ; (b), (c), (d) Triangle structure, with the apex angle α computation illustrated.

Turning our attention to h_{max} and Figure 5(c), we find that h_{max} occurs with $\triangle wvZ$ having apex angle of $\theta - \delta$ and base angle of $\beta + \delta$. Let $2y$ be the amount by which the base $2L$ is shortened. Using $\triangle wAa$ we find that $\sin(\beta+\delta) = \frac{\epsilon}{y}$. Using $\triangle wvZ$ we find that $\tan(\beta+\delta) = \frac{h_{max}}{L-y}$. Thus we have

$$\begin{aligned} h_{max} &= \tan(\beta+\delta)(L-y) = \tan(\beta+\delta)\left(L - \frac{\epsilon}{\sin(\beta+\delta)}\right) \\ &= L\tan(\beta+\delta) - \frac{\epsilon}{\cos(\beta+\delta)}. \end{aligned}$$

So h_{min} and h_{max} are continuous near $\epsilon = 0$. Also, if $\epsilon \to 0$ then $\delta \to 0$ since $\delta \le \frac{2\epsilon}{x}$. Therefore $h_{min} = \lim_{\epsilon\to 0} L\tan\beta = h$, since $\tan\beta = h/L$. In the same manner we have that $\lim_{\epsilon\to 0} h_{max} = L\tan\beta = h$. □

3.4. **Main Theorem.** We connect 3D to 2D via the plane determined by the 2-chain in the proof of the main theorem below.

Theorem 1. *The 2-link chain is interlocked with the 11-link triangle chain.*

Proof. Let H be the plane containing the 2-link chain. The links of the 2-chain pass through ϵ-balls around the three corners of the triangle.

H meets these balls in disks each of radius $\leq \epsilon$. The Triangle Lemma 2 shows that the height of the triangle approaches h as ϵ approaches 0. Thus, by choosing ϵ small enough, we limit the amount that the apex v of the 2-link chain can be separated from or pushed toward the triangle to any desired amount.

Similar to [5], and using Lemma 1 we can establish that the 2-chain links are interlocked with the 3/4-tangle and jag loops through which they pass, under the assumption that the triangle is nearly rigid.

Thus, choosing ϵ small enough to prevent the two end vertices of the 2-link chain from entering the ϵ-balls, ensures that the 2-link chain is interlocked with the triangle chain. □

4. Discussion

The obvious open problem is to prove or disprove that $k = 11$ is minimal. We suspect that $k = 11$ is minimal because, on the one hand, it seems a 3/4-tangle must be employed in the construction, and on the other hand, it seems impossible to employ the tangle more efficiently than realized in Figure 3. For example, to achieve a 10-chain, it would be necessary to arrange for maximal sharing of the 3/4-tangle's 7 links, using two of its links to serve as two sides of the triangle frame. This maximum sharing would require near parallelism of the links va and yz, and near parallelism of the links vb and DE. But this cannot be achieved, for the following reason.

From the fact that the convex hull $CH(B, C, D, x, y)$ does not change its combinatorial structure (Fact (1)), and the result that the 2-chain (a, v, b) is interlocked with the 11-chain, the convex hull $CH(B, C, D, v, b)$ does not change either. Because link va penetrates the plane BCD the same as link yz (cf. Figure 4), and the link vb penetrates the plane BCD the same as link yx, it follows that both links DE and AB penetrate the plane of the 2-chain (a, v, b). Hence neither DE nor AB can be arranged to be nearly parallel to the 2-chain link vb, which would be necessary for maximal sharing.

Thus, it seems that if $k = 11$ is not minimal, it would require a fundamentally different construction, which seems difficult because there is no 3/3 interlocked tangle [3, Thm. 1].

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