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# $\pi/2$-ANGLE YAO GRAPHS ARE SPANNERS

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We show that the Yao graph $Y_4$ in the $L_2$ metric is a spanner with stretch factor $8\sqrt{2}(26+23\sqrt{2})$. Enroute to this, we also show that the Yao graph $Y_4^\infty$ in the $L_\infty$ metric is a plane spanner with stretch factor 8.

*Keywords*: Yao graph; Y4; spanner.

## 1. Introduction

Let $V$ be a finite set of points in the plane and let $G = (V, E)$ be the complete Euclidean graph on $V$. We will refer to the points in $V$ as *nodes*, to distinguish them from other points in the plane. The *Yao graph* [8] with an integer parameter $k > 0$, denoted $Y_k$, is defined as follows. Any $k$ equally-separated rays starting at the origin define $k$ cones. Pick a set of arbitrary, but fixed cones. Translate the cones to each node $u \in V$. In each cone with apex $u$, pick a shortest edge $uv$, if there is one, and add to $Y_k$ the directed edge $\overrightarrow{uv}$. Ties are broken arbitrarily. Note that the Yao graph differs from the $\Theta$-graph in how the shortest edge is chosen. While the Yao graph chooses the shortest edge in terms of the Euclidean distance, the $\Theta$-graph chooses the edge whose projection on the bisector of the cone is shortest. Most of the time we ignore the direction of an edge $uv$; we refer to the directed version $\overrightarrow{uv}$ of $uv$ only when its origin ($u$) is important and unclear from the context. We will distinguish between $Y_k$, the Yao graph in the Euclidean $L_2$ metric, and $Y_k^\infty$, the Yao graph in the $L_\infty$ metric. Unlike $Y_k$ however, in constructing $Y_k^\infty$ ties are broken by always selecting the most counterclockwise edge; the reason for this choice will become clear in Section 2.

The *length* of a path is the sum of the lengths of its constituent edges. For a given subgraph $H \subseteq G$ and a fixed $t \geq 1$, $H$ is called a *t-spanner* for $G$ if, for any two nodes $u, v \in V$, the shortest path in $H$ from $u$ to $v$ is no longer than $t$ times the length $|uv|$ of $uv$. The value $t$ is called the *dilation* or the *stretch factor* of $H$. If $t$ is constant, then $H$ is called a *length spanner*, or simply a *spanner*.

The class of graphs $Y_k$ has been much studied. Bose et al. [2] showed that, for $k \geq 9$, $Y_k$ is a spanner with stretch factor $\frac{1}{\cos\frac{2\pi}{k} - \sin\frac{2\pi}{k}}$. In Ref. [1] we improved the stretch factor and showed that, in fact, $Y_k$ is a spanner for any $k \geq 7$. Recently, Damian and Raudonis [4] showed that $Y_6$ is a 17.7-spanner. Molla [6] showed that $Y_2$ and $Y_3$ are not spanners, and that $Y_4$ is a spanner with stretch factor $4(2 + \sqrt{2})$, for the special case when the nodes in $V$ are in convex position (see also Ref. [3]). The authors conjectured that $Y_4$ is a spanner for arbitrary point sets. In this paper, we settle their conjecture and prove that $Y_4$ is a spanner with stretch factor $8\sqrt{2}(26 + 23\sqrt{2})$.

The paper is organized as follows. In Section 2, we prove that the graph $Y_4^\infty$ is a spanner with stretch factor 8. In Section 3 we establish several properties for the graph $Y_4$. Finally, in Section 4, we use the properties of Section 3 to prove that, for every edge $ab$ in $Y_4^\infty$, there exists a path between $a$ and $b$ in $Y_4$ not much longer than the Euclidean distance between $a$ and $b$. By combining this with the result of Section 2, we conclude that $Y_4$ is a spanner.

## 2. $Y_4^\infty$ in the $L_\infty$ Metric

In this section we focus on $Y_4^\infty$, which has a nicer structure compared to $Y_4$. First we prove that $Y_4^\infty$ is a plane graph. Then we use this property to show that $Y_4^\infty$ is an 8-spanner. To be more precise, we prove that for any two nodes $a$ and $b$, the graph $Y_4^\infty$ contains a path between $a$ and $b$ whose length (in the $L_\infty$-metric) is at most $8|ab|_\infty$.

We need a few definitions. We say that two edges $ab$ and $cd$ *properly cross* (or *cross*, for short) if they share a point other than an endpoint ($a$, $b$, $c$ or $d$); we say that $ab$ and $cd$ *intersect* if they share a point (either an interior point or an endpoint).

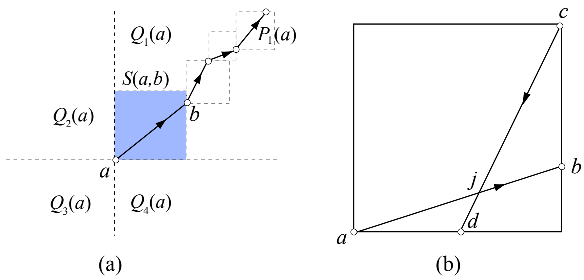


Fig. 1. (a) Definitions: $Q_i(a)$, $P_i(a)$ and $S(a, b)$. (b) Lemma 1: $ab$ and $cd$ cannot cross.

Throughout the paper, we will use the following notation: for each node $a \in V$, $x(a)$ is the $x$-coordinate of $a$ and $y(a)$ is the $y$-coordinate of $a$; $Q_1(a)$, $Q_2(a)$, $Q_3(a)$ and $Q_4(a)$ are the four quadrants at $a$, depicted in Fig. 1a; each quadrant is half-open and half-closed, including all points on the clockwise boundary axis (with respect to the quadrant bisector through $a$), and excluding all points on the counterclockwise boundary axis; $P_i(a)$ is the path that starts at $a$ and follows the directed Yao edges in quadrant $Q_i$; $P_i(a, b)$ is the subpath of $P_i(a)$ that starts at node $a$ and ends at node $b$; $|ab|_\infty$ is the $L_\infty$ distance between $a$ and $b$, defined as $\max\{|x(a)-x(b)|, |y(a)-y(b)|\}$; $sp(a, b)$ is a shortest path in $Y_4^\infty$ between $a$ and $b$; $S(a, b)$ is the open square with corner $a$ whose boundary contains $b$; and $\partial S(a, b)$ is the boundary of $S(a, b)$. These definitions are depicted in Fig. 1a.

**Lemma 1.** *$Y_4^\infty$ is a plane graph.*

**Proof.** The proof is by contradiction. Assume the opposite. Then there are two edges $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4^\infty$ that cross each other. Since $\overrightarrow{ab} \in Y_4^\infty$, $S(a, b)$ must be empty of nodes in $V$, and similarly for $S(c, d)$. Let $j$ be the intersection point between $ab$ and $cd$. Then $j \in S(a, b) \cap S(c, d)$, meaning that $S(a, b)$ and $S(c, d)$ must overlap. However, neither square may contain $a, b, c$ or $d$. It follows that $S(a, b)$ and $S(c, d)$ coincide, meaning that $c$ and $d$ lie on $\partial S(a, b)$ (see Fig. 1b). Since $cd$ intersects $ab$, $c$ and $d$ must lie on opposite sides of $ab$. Thus either $ac$ or $ad$ lies counterclockwise from $ab$. Assume without loss of generality that $ac$ lies counterclockwise from $ab$; the other case is identical. Because $S(a, c)$ coincides with $S(a, b)$, we have that $|ac|_\infty = |ab|_\infty$. In this case however, $Y_4^\infty$ would break the tie between $ac$ and $ab$ by selecting the most counterclockwise edge, which is $\overrightarrow{ac}$. This contradicts that $\overrightarrow{ab} \in Y_4^\infty$. □

**Theorem 1.** *$Y_4^\infty$ is an 8-spanner in the $L_\infty$ metric.*

**Proof.** We show that, for any pair of points $a, b \in V$, $|sp(a,b)|_\infty < 8|ab|_\infty$. The proof is by induction on the pairwise $L_\infty$-distance between the points in $V$. Assume without loss of generality that $b \in Q_1(a)$, and $|ab|_\infty = |x(b) - x(a)|$ (i.e., $b$ lies below the diagonal of $S(a,b)$ incident to $a$). Consider the case in which $ab$ is a closest (in the $L_\infty$ metric) pair of points in $V$. This is the base case for our induction. If $ab \in Y_4^\infty$, then $|sp(a,b)|_\infty = |ab|_\infty$. Otherwise, there must be $ac \in Y_4^\infty$, with $|ac|_\infty = |ab|_\infty$. Recall that $Y_4^\infty$ breaks ties by always selecting the most counterclockwise edge, so $ac$ must be counterclockwise of $ab$. Also recall that $Q_1(a)$ does not include the vertical coordinate axis through $a$, therefore $c$ lies strictly to the right of $a$. It follows that $|bc|_\infty < |ab|_\infty$ (see Fig. 2a), a contradiction.

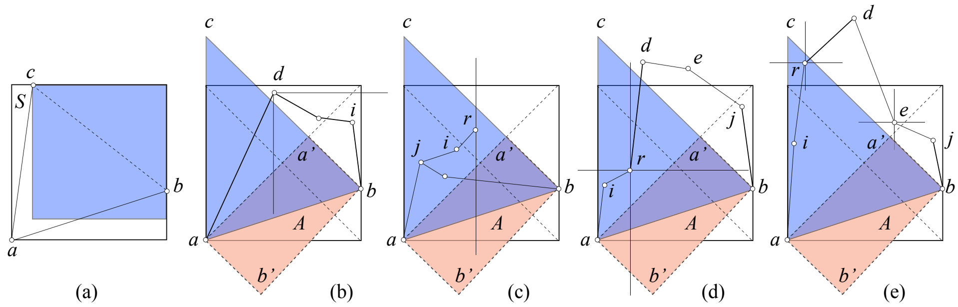


Fig. 2. (a) Base case. (b) $\triangle abc$ empty (c) $\triangle abc$ non-empty, $P_{ar} \cap P_2(b) = \{j\}$ (d) $\triangle abc$ non-empty, $P_{ar} \cap P_2(b) = \emptyset$, $e$ above $r$ (e) $\triangle abc$ non-empty, $P_{ar} \cap P_2(b) = \emptyset$, $e$ below $r$.

Assume now that the inductive hypothesis holds for all pairs of points closer (in the $L_\infty$ metric) than $|ab|_\infty$. If $ab \in Y_4^\infty$, then $|sp(a,b)|_\infty = |ab|_\infty$ and the proof is finished. If $ab \notin Y_4^\infty$, then the square $S(a,b)$ must be nonempty.

Let $A$ be the rectangle $ab'ba'$ as in Fig. 2b, where $ba'$ and $bb'$ are parallel to the diagonals of $S(a,b)$. If $A$ is nonempty, then we can use induction to prove that $|sp(a,b)|_\infty \le 8|ab|_\infty$ as follows. Pick $c \in A$ arbitrary. Then $|ac|_\infty + |cb|_\infty = |x(c) - x(a)| + |x(b) - x(c)| = |ab|_\infty$, and by the inductive hypothesis $sp(a,c) \oplus sp(c,b)$ is a path in $Y_4^\infty$ no longer than $8|ac|_\infty + 8|cb|_\infty = 8|ab|_\infty$; here $\oplus$ represents the concatenation operator. Assume now that $A$ is empty. Let $c$ be at the intersection between the line supporting $ba'$ and the vertical line through $a$ (see Fig. 2b). We discuss two cases, depending on whether $\triangle abc$ is empty of points or not.

**Case 1:** $\triangle abc$ is empty of points. Let $ad \in P_1(a)$. We show that $P_4(d)$ cannot contain an edge crossing $ab$. Assume the opposite, and let $st \in P_4(d)$ cross $ab$. Note that $st \in P_4(d)$ also implies $st \in P_4(s)$, which along with the fact that $st$ crosses $ab$, implies that $s$ is either vertically aligned, or to the left of $b$.. Since $\triangle abc$ is empty, $s$ must lie above $bc$ and $t$ below $ab$. It follows that $b$ and $t$ are in the same quadrant

$Q_4(s)$ (recall that this quadrant includes the downward ray from $s$). Furthermore, $|st|_\infty \geq |y(s) - y(t)| > |y(s) - y(b)| = |sb|_\infty$, contradicting the fact that $st \in Y_4^\infty$.

We have established that $P_4(d)$ does not cross $ab$, which implies that $P_4(d)$ must exit $S(d, b)$ through its right edge. Also note that $P_2(b)$ cannot cross $ac$, because $\triangle abc$ is empty of points, and any point left of $ac$ is $L_\infty$-farther from $b$ than $d$. It follows that $P_2(b)$ exits $S(b, d)$ through its top edge. This together with the fact that $P_4(d)$ exits $S(d, b)$ through its right edge, implies that $P_4(d)$ and $P_2(b)$ must meet in a point $i \in P_4(d) \cap P_2(b)$ (see Fig. 2b). Now note that $|P_4(d, i) \oplus P_2(b, i)|_\infty \leq |x(d) - x(b)| + |y(d) - y(b)| < 2|ab|_\infty$. Thus we have that $|sp(a, b)|_\infty \leq |ad \oplus P_4(d, i) \oplus P_2(b, i)|_\infty < |ab|_\infty + 2|ab|_\infty = 3|ab|_\infty$.

**Case 2:** $\triangle abc$ is nonempty. In this case, we seek a short path from $a$ to $b$ that does not cross to the underside of $ab$, to avoid oscillating paths that cross $ab$ arbitrarily many times. Let $r$ be the rightmost point that lies inside $\triangle abc$. Arguments similar to the ones used in Case 1 show that $P_3(r)$ cannot cross $ab$ and therefore it must meet $P_1(a)$ in a point $i$. Then $P_{ar} = P_1(a, i) \oplus P_3(r, i)$ is a path in $Y_4^\infty$ of length

$$|P_{ar}|_\infty < |x(a) - x(r)| + |y(a) - y(r)| < |ab|_\infty + 2|ab|_\infty = 3|ab|_\infty. \qquad (1)$$

The term $2|ab|_\infty$ in the inequality above results from the fact that $|y(a) - y(r)| \leq |y(a) - y(c)| \leq 2|ab|_\infty$. Consider first the simpler situation in which $P_2(b)$ meets $P_{ar}$ in a point $j \in P_2(b) \cap P_{ar}$ (see Fig. 2c). Let $P_{ar}(a, j)$ be the subpath of $P_{ar}$ extending between $a$ and $j$. Then $P_{ar}(a, j) \oplus P_2(b, j)$ is a path in $Y_4^\infty$ from $a$ to $b$, therefore $|sp(a, b)|_\infty \leq |P_{ar}(a, j) \oplus P_2(b, j)|_\infty < 2|y(j) - y(a)| + |ab|_\infty \leq 5|ab|_\infty$.

Consider now the case when $P_2(b)$ does not intersect $P_{ar}$. We argue that, in this case, $Q_1(r)$ may not be empty. Assume the opposite. Then no edge $st \in P_2(b)$ may cross $Q_1(r)$. This is because, for any such edge, $|sr|_\infty < |st|_\infty$, contradicting $st \in Y_4^\infty$. This implies that $P_2(b)$ intersects $P_{ar}$, again a contradiction to our assumption. This establishes that $Q_1(r)$ is nonempty. Let $rd \in P_1(r)$. The fact that $P_2(b)$ does not intersect $P_{ar}$ implies that $d$ lies to the left of $b$. The fact that $r$ is the rightmost point in $\triangle abc$ implies that $d$ lies outside $\triangle abc$ (see Fig. 2d). It also implies that $P_4(d)$ shares no points with $\triangle abc$. This along with arguments similar to the ones used in case 1 show that $P_4(d)$ and $P_2(b)$ meet in a point $j \in P_4(d) \cap P_2(b)$. Thus we have found a path

$$P_{ab} = P_1(a, i) \oplus P_3(r, i) \oplus rd \oplus P_4(d, j) \oplus P_2(b, j). \qquad (2)$$

extending from $a$ to $b$ in $Y_4^\infty$. If $|rd|_\infty = |x(d) - x(r)|$, then $|rd|_\infty < |x(b) - x(a)| = |ab|_\infty$, and the path $P_{ab}$ has length

$$|P_{ab}|_\infty \leq 2|y(d) - y(a)| + |ab|_\infty < 7|ab|_\infty. \qquad (3)$$

In the above, we used the fact that $|y(d) - y(a)| = |y(d) - y(r)| + |y(r) - y(a)| < |ab|_\infty + 2|ab|_\infty$ . Suppose now that

$$|rd|_\infty = |y(d) - y(r)|. \qquad (4)$$

In this case, it is unclear whether the path $P_{ab}$ defined by (2) is short, since $rd$ can be arbitrarily long compared to $ab$. Let $e$ be the clockwise neighbor of $d$ along the path $P_{ab}$ ($e$ and $b$ may coincide). Then $e$ lies below $d$, and either $de \in P_4(d)$, or $ed \in P_2(e)$ (or both). If $e$ lies above $r$, or at the same level as $r$ (i.e., $e \in Q_1(r)$, as in Fig. 2d), then

$$|y(e) - y(r)| < |y(d) - y(r)|. \tag{5}$$

Since $rd \in P_1(r)$ and $e$ is in the same quadrant of $r$ as $d$, we have $|rd|_\infty \leq |re|_\infty$. This along with inequalities (4) and (5) implies $|re|_\infty > |y(e) - y(r)|$, which in turn implies $|re|_\infty = |x(e) - x(r)| \leq |ab|_\infty$, and so $|rd|_\infty \leq |ab|_\infty$. Then inequality (3) applies here as well, showing that $|P_{ab}|_\infty < 7|ab|_\infty$.

If $e$ lies below $r$ (as in Fig. 2e), then

$$|ed|_\infty \geq |y(d) - y(e)| \geq |y(d) - y(r)| = |rd|_\infty. \tag{6}$$

Assume first that $ed \in P_2(e)$, or $|ed|_\infty = |x(e) - x(d)|$. In either case, $|ed|_\infty \leq |er|_\infty < 2|ab|_\infty$. This along with inequality (6) shows that $|rd|_\infty < 2|ab|_\infty$. Substituting this upper bound in (2), we get $|P_{ab}|_\infty \leq 2|y(d) - y(a)| + 2|ab|_\infty < 8|ab|_\infty$. Assume now that $ed \notin P_2(e)$, and $|ed|_\infty = |y(e) - y(d)|$. Then $ee' \in P_2(e)$ cannot go above $d$ (otherwise $|ed|_\infty < |ee'|_\infty$, contradicting $ee' \in P_2(e)$). This along with the fact $de \in P_4(d)$ implies that $P_2(e)$ intersects $P_{ar}$ in a point $k$. Redefine $P_{ab} = P_{ar}(a,k) \oplus P_2(e,k) \oplus P_4(e,j) \oplus P_2(b,j)$. Then $P_{ab}$ is a path in $Y_4^\infty$ from $a$ to $b$ of length $|P_{ab}| \leq 2|y(r) - y(a)| + |ab|_\infty \leq 5|ab|_\infty$. □

This theorem will be employed in Section 4.

## 3. $Y_4$ in the $L_2$ Metric

In this section we establish basic properties of $Y_4$. The ultimate goal of this section is to show that, if two edges in $Y_4$ cross, there is a short path between their endpoints (Lemma 8). We begin with a few definitions.

Let $Q(a,b)$ denote the infinite quadrant with origin at $a$ that contains $b$. For a pair of nodes $a, b \in V$, define recursively a directed path $\mathcal{P}(a \to b)$ from $a$ to $b$ in $Y_4$ as follows. If $a = b$, then $\mathcal{P}(a \to b) = null$. If $a \neq b$, there must exist $\overrightarrow{ac} \in Y_4$ that lies in $Q(a,b)$. In this case, define

$$\mathcal{P}(a \to b) = \overrightarrow{ac} \oplus \mathcal{P}(c \to b).$$

Recall that $\oplus$ represents the concatenation operator. This definition is illustrated in Fig. 3a. Fischer et al. [5] show that $\mathcal{P}(a \to b)$ is well defined and lies entirely inside the square centered at $b$ whose boundary contains $a$.

For any path $P$ and any pair of nodes $a, b \in P$, let $P[a,b]$ be the subpath of $P$ from $a$ to $b$. Let $R(a,b)$ be the closed axis-aligned rectangle with diagonal $ab$ "(we permit $R(a,b)$ to be degenerate rectangle, when $ab$ is either horizontal or vertical).

For a fixed pair of nodes $a, b \in V$, define a path $\mathcal{P}_R(a \to b)$ as follows. Let $e \in V$ be the first node along $\mathcal{P}(a \to b)$ that is not strictly interior to $R(a,b)$. Then

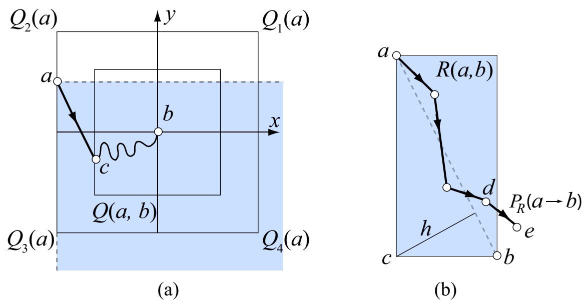


Fig. 3. Definitions. (a) $Q(a, b)$ and $\mathcal{P}(a \to b)$. (b) $\mathcal{P}_R(a \to b)$.

$\mathcal{P}_R(a \to b)$ is the subpath of $\mathcal{P}(a \to b)$ that extends between $a$ and $e$. In other words, $\mathcal{P}_R(a \to b)$ is the path that follows the $Y_4$ edges pointing towards $b$, truncated as soon as it reaches $b$ or leaves $R(a, b)$. Formally, $\mathcal{P}_R(a \to b) = \mathcal{P}(a \to b)[a, e]$. This definition is illustrated in Fig. 3b. Our proofs will make use of the following two propositions.

**Proposition 1.** *The sum of the lengths of crossing diagonals of a non-degenerate (necessarily convex) quadrilateral abcd is strictly greater than the sum of the lengths of either pair of opposite sides:*

$$|ac| + |bd| > |ab| + |cd|$$
$$|ac| + |bd| > |bc| + |da|.$$

This can be proved by partitioning the diagonals into two pieces each at their intersection point, and then applying the triangle inequality twice.

**Proposition 2.** *For any triangle $\triangle abc$, the following inequalities hold:*

$$|ac|^2 \begin{cases} < |ab|^2 + |bc|^2, & \text{if } \angle abc < \pi/2 \\ = |ab|^2 + |bc|^2, & \text{if } \angle abc = \pi/2 \\ > |ab|^2 + |bc|^2, & \text{if } \angle abc > \pi/2 \end{cases}$$

This proposition follows immediately from the Law of Cosines applied to triangle $\triangle abc$.

**Lemma 2.** *For each pair of nodes $a, b \in V$,*

$$|\mathcal{P}_R(a \to b)| \le |ab|\sqrt{2}. \tag{7}$$

*Furthermore, each edge of $\mathcal{P}_R(a \to b)$ is no longer than $|ab|$.*

**Proof.** Let $c$ be one of the two corners of $R(a, b)$, other than $a$ and $b$. Let $\overrightarrow{de} \in \mathcal{P}_R(a \to b)$ be the last edge on $\mathcal{P}_R(a \to b)$, which necessarily intersects $\partial R(a, b)$

(note that it is possible that $e = b$). Refer to Fig. 3b. Then $|de| \leq |db|$, otherwise $\overrightarrow{de}$ could not be in $Y_4$. Since $db$ lies in the rectangle with diagonal $ab$, we have that $|db| \leq |ab|$, and similarly for each edge on $\mathcal{P}_R(a \to b)$. This establishes the latter claim of the lemma. For the first claim of the lemma, let $p = \mathcal{P}_R(a \to b)[a, d] \oplus db$. Since $|de| \leq |db|$, we have that $|\mathcal{P}_R(a \to b)| \leq |p|$. Since $p$ lies entirely inside $R(a, b)$ and consists of edges pointing towards $b$, we have that $p$ is an $xy$-monotone path (i.e., any line parallel to a coordinate axis intersects $p$ in at most one point). It follows that $|p| \leq |ac| + |cb|$, which is bounded above by $|ab|\sqrt{2}$. □

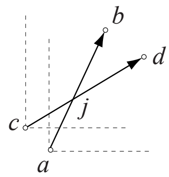


Fig. 4. Lemma 3: if $ab$ and $cd$ cross, they cannot both be in $Y_4$.

**Lemma 3.** *Let $a, b, c, d \in V$ be four disjoint nodes such that $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4$, $b \in Q_i(a)$ and $d \in Q_i(c)$, for some $i \in \{1, 2, 3, 4\}$. Then $ab$ and $cd$ cannot cross.*

**Proof.** We may assume without loss of generality that $i = 1$ and $c$ is to the left of $a$. The proof is by contradiction. Assume that $ab$ and $cd$ cross each other. Let $j$ be the intersection point between $ab$ and $cd$ (see Fig. 4). Since $j \in Q_1(a) \cap Q_1(c)$, it follows that $d \in Q_1(a)$ and $b \in Q_1(c)$. Thus $|ab| \leq |ad|$, because otherwise, $\overrightarrow{ab}$ cannot be in $Y_4$. By Proposition 1 applied to the quadrilateral $adbc$,

$$|ad| + |cb| < |ab| + |cd|.$$

This along with $|ab| \leq |ad|$ implies that $|cb| < |cd|$, contradicting that $\overrightarrow{cd} \in Y_4$. □

The next four lemmas (4–7) each concern a pair of crossing $Y_4$ edges, culminating (in Lemma 8) in the conclusion that there is a short path in $Y_4$ between a pair of endpoints of those edges. We choose to defer the proofs of lemmas 4–6 to the appendix, for a better understanding of the logical flow of our analysis.

**Lemma 4.** *Let $a$, $b$, $c$ and $d$ be four disjoint nodes in $V$ such that $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4$, and $ab$ crosses $cd$. Then (i) the ratio between the shortest side and the longer diagonal of the quadrilateral $acbd$ is no greater than $1/\sqrt{2}$, and (ii) the shortest side of the quadrilateral $acbd$ is strictly shorter than either diagonal.*

**Lemma 5.** *Let $a, b, c, d$ be four distinct nodes in $V$, with $c \in Q_1(a)$, such that (i) $\overrightarrow{ab} \in Q_1(a)$ and $\overrightarrow{cd} \in Q_2(c)$ are in $Y_4$ and cross each other, and (ii) $ad$ is a shortest* *side of quadrilateral acbd. Then* $\mathcal{P}_R(a \to d)$ *and* $\mathcal{P}_R(d \to a)$ *have a nonempty intersection.*

**Lemma 6.** *Let* $a, b, c, d$ *be four distinct nodes in* $V$*, with* $c \in Q_1(a)$*, such that (i)* $\overrightarrow{ab} \in Q_1(a)$ *and* $\overrightarrow{cd} \in Q_3(c)$ *are in* $Y_4$ *and cross each other, and (ii) ad is a shortest side of quadrilateral acbd. Then* $\mathcal{P}_R(d \to a)$ *does not cross ab.*

The next lemma relies on all of Lemmas 2–6.

**Lemma 7.** *Let* $a, b, c, d \in V$ *be four distinct nodes such that* $\overrightarrow{ab} \in Y_4$ *crosses* $\overrightarrow{cd} \in Y_4$*, and let xy be a shortest side of the quadrilateral acbd. Then there exist two paths* $\mathcal{P}_x$ *and* $\mathcal{P}_y$ *in* $Y_4$*, where* $\mathcal{P}_x$ *has* $x$ *as an endpoint and* $\mathcal{P}_y$ *has* $y$ *as an endpoint, with the following properties:*

*(i)* $\mathcal{P}_x$ *and* $\mathcal{P}_y$ *have a nonempty intersection.*
*(ii)* $|\mathcal{P}_x| + |\mathcal{P}_y| \leq 3\sqrt{2}|xy|$.
*(iii) Each edge on* $\mathcal{P}_x \cup \mathcal{P}_y$ *is no longer than* $|xy|$.

**Proof.** Assume without loss of generality that $b \in Q_1(a)$. We discuss the following exhaustive cases:

(1) $c \in Q_1(a)$, and $d \in Q_1(c)$. In this case, $ab$ and $cd$ cannot cross each other (by Lemma 3), so this case is finished.

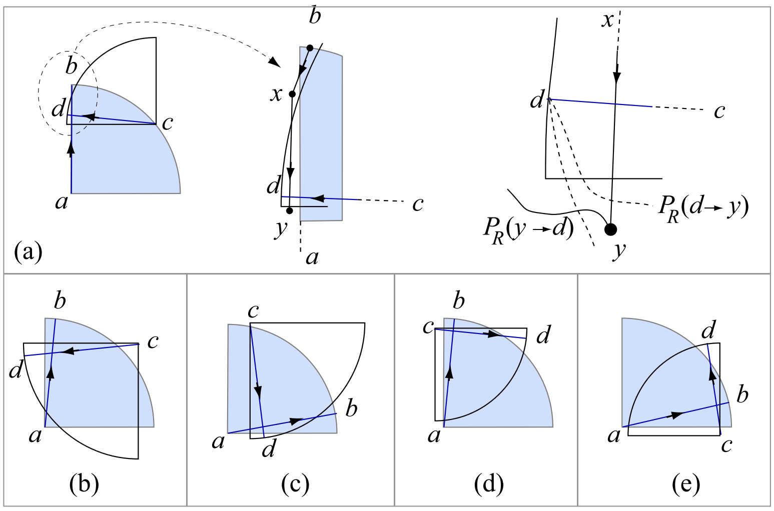


Fig. 5. Lemma 7: (a, b) $c \in Q_1(a)$ (c) $c \in Q_2(a)$ (d) $c \in Q_4(a)$.

(2) $c \in Q_1(a)$, and $d \in Q_2(c)$, as in Fig. 5a. Since $\overrightarrow{ab} \in Y_4$, $|ab| \leq |ac|$. Since $ab$ crosses $cd$, and $|ab| \leq |ac|$, $b \in Q_2(c)$. Since $\overrightarrow{cd} \in Y_4$, $|cd| \leq |cb|$. These along

with Lemma 4 imply that $ad$ and $db$ are the only candidates for a shortest edge of $acbd$. Assume first that $ad$ is a shortest edge of $acbd$. By Lemma 3, $\mathcal{P}_a = \mathcal{P}_R(a \to d)$ does not cross $cd$, because $\mathcal{P}_a \in Q_2(a)$ and $cd \in Q_2(c)$ are in the quadrants of identical indices. It follows from Lemma 5 that $\mathcal{P}_a$ and $\mathcal{P}_d = \mathcal{P}_R(d \to a)$ have a nonempty intersection. Furthermore, by Lemma 2, $|\mathcal{P}_a| \le |ad|\sqrt{2}$ and $|\mathcal{P}_d| \le |ad|\sqrt{2}$, and no edge on these paths is longer than $|ad|$, proving the lemma true for this case. Consider now the case when $db$ is a shortest edge of $acbd$ (see Fig. 5a). Note that $d$ is below $b$ (otherwise, $d \in Q_2(c)$ and $|cd| > |cb|$) and, therefore, $b \in Q_1(d)$. By Lemma 3, $\mathcal{P}_d = \mathcal{P}_R(d \to b)$ does not cross $ab$, because $\mathcal{P}_d \in Q_1(d)$ and $ab \in Q_1(a)$. If $\mathcal{P}_b = \mathcal{P}_R(b \to d)$ does not cross $cd$, then $\mathcal{P}_b$ and $\mathcal{P}_d$ have a nonempty intersection, proving the lemma true for this case. Otherwise, there exists $\overrightarrow{xy} \in \mathcal{P}_R(b \to d)$ that crosses $cd$ (see Fig. 5a). Define

$$\mathcal{P}_b = \mathcal{P}_R(b \to d) \oplus \mathcal{P}_R(y \to d)$$
$$\mathcal{P}_d = \mathcal{P}_R(d \to y).$$

By Lemma 3, $\mathcal{P}_R(y \to d)$ does not cross $cd$, because they are both in quadrant $Q_2$. Then $\mathcal{P}_b$ and $\mathcal{P}_d$ must have a nonempty intersection. We now show that $\mathcal{P}_b$ and $\mathcal{P}_d$ satisfy conditions (i) and (iii) of the lemma. Proposition 1 applied on the quadrilateral $xdyc$ tells us that $|xc| + |yd| < |xy| + |cd|$. We also have that $|cx| \ge |cd|$, since $\overrightarrow{cd} \in Y_4$ and $x$ is in the same quadrant of $c$ as $d$. This along with the inequality above implies $|yd| < |xy|$. Because $xy \in \mathcal{P}_R(b \to d)$, by Lemma 2 we have that $|xy| \le |bd|$, which along with the previous inequality shows that $|yd| < |bd|$. This along with Lemma 2 shows that condition (iii) of the lemma is satisfied. Furthermore, $|\mathcal{P}_R(y \to d)| \le |yd|\sqrt{2}$ and $|\mathcal{P}_R(d \to y)| \le |yd|\sqrt{2}$. It follows that $|\mathcal{P}_b| + |\mathcal{P}_d| \le 3\sqrt{2}|bd|$.

(3) $c \in Q_1(a)$, and $d \in Q_3(c)$, as in Fig. 5b. Then $|ac| \ge \max\{ab, cd\}$, and by Lemma 4 $ac$ is not a shortest edge of $acbd$. The case when $bd$ is a shortest edge of $acbd$ is settled by Lemmas 3 and 2: Lemma 3 tells us that $\mathcal{P}_d = \mathcal{P}_R(d \to b)$ does not cross $ab$, (because they are both in $Q_1$,) and $\mathcal{P}_b = \mathcal{P}_R(b \to d)$ does not cross $cd$ (because they are both in $Q_3$). It follows that $\mathcal{P}_d$ and $\mathcal{P}_b$ have a nonempty intersection. Furthermore, Lemma 2 guarantees that $\mathcal{P}_d$ and $\mathcal{P}_b$ satisfy conditions (ii) and (iii) of the lemma. Consider now the case when $ad$ is a shortest edge of $acbd$; the case when $bc$ is shortest is symmetric. By Lemma 6, $\mathcal{P}_R(d \to a)$ does not cross $ab$. If $\mathcal{P}_R(a \to d)$ does not cross $cd$, then this case is settled: $\mathcal{P}_d = \mathcal{P}_R(d \to a)$ and $\mathcal{P}_a = \mathcal{P}_R(a \to d)$ satisfy the three conditions of the lemma. Otherwise, let $\overrightarrow{xy} \in \mathcal{P}_R(a \to d)$ be the edge crossing $cd$. Arguments similar to the ones used in case 1 above show that $\mathcal{P}_a = \mathcal{P}_R(a \to d) \oplus \mathcal{P}_R(y \to d)$ and $\mathcal{P}_d = \mathcal{P}_R(d \to y)$ are two paths that satisfy the conditions of the lemma.

(4) $c \in Q_1(a)$, and $d \in Q_4(c)$, as in Fig. 5c. Note that a horizontal reflection of Fig. 5c, followed by a rotation of $\pi/2$, depicts a case identical to case (2), Fig. 5a, which has already been settled.

(5) $c \in Q_2(a)$, as in Fig. 5d. Note that Fig. 5d rotated by $\pi/2$ depicts a case identical to case (2), Fig. 5a (with the roles of $ab$ and $cd$ switched), which has already been settled.

(6) $c \in Q_3(a)$. Then it must be that $d \in Q_1(c)$, otherwise $cd$ cannot cross $ab$. By Lemma 3 however, $ab$ and $cd$ may not cross, unless one of them is not in $Y_4$.

(7) $c \in Q_4(a)$. By Lemma 3, $d$ may not lie in $Q_1(c)$, therefore $d$ must be in $Q_2(c)$, as in Fig. 5e. Note that a vertical reflection of Fig. 5e depicts a case identical to case (2), Fig. 5a (with the roles of $ab$ and $cd$ switched), so this case is settled as well. □

We are now ready to establish the main lemma of this section, showing that there is a short path between the endpoints of two intersecting edges in $Y_4$.

**Lemma 8.** *Let $a, b, c, d \in V$ be four distinct nodes such that $\overrightarrow{ab} \in Y_4$ crosses $\overrightarrow{cd} \in Y_4$, and let $xy$ be a shortest side of the quadrilateral $acbd$. Then $Y_4$ contains a path $p(x, y)$ connecting $x$ and $y$, of length $|p(x, y)| \leq \frac{6}{\sqrt{2}-1} \cdot |xy|$. Furthermore, no edge on $p(x, y)$ is longer than $|xy|$.*

**Proof.** Let $\mathcal{P}_x$ and $\mathcal{P}_y$ be the two paths whose existence in $Y_4$ is guaranteed by Lemma 7. By condition (iii) of Lemma 7, no edge on $\mathcal{P}_x$ and $\mathcal{P}_y$ is longer than $|xy|$. By condition (i) of Lemma 7, $\mathcal{P}_x$ and $\mathcal{P}_y$ have a nonempty intersection. If $\mathcal{P}_x$ and $\mathcal{P}_y$ share a node $u \in V$, then the path $p(x, y) = \mathcal{P}_x[x, u] \oplus \mathcal{P}_y[y, u]$ is a path from $x$ to $y$ in $Y_4$ no longer than $3\sqrt{2}|xy|$; the length restriction follows from guarantee (ii) of Lemma 7. Otherwise, let $\overrightarrow{a'b'} \in \mathcal{P}_x$ and $\overrightarrow{c'd'} \in \mathcal{P}_y$ be two edges crossing each other. Let $x'y'$ be a shortest side of the quadrilateral $a'c'b'd'$, with $x' \in \mathcal{P}_x$ and $y' \in \mathcal{P}_y$. Lemma 7 tells us that $|a'b'| \leq |xy|$ and $|c'd'| \leq |xy|$. These along with Lemma 4 imply that

$$|x'y'| \leq |xy|/\sqrt{2}. \tag{8}$$

This enables us to derive a recursive formula for computing a path $p(x, y) \in Y_4$ as follows:

$$p(x, y) = \begin{cases} x, & \text{if } x = y \\ \mathcal{P}_x[x, x'] \oplus \mathcal{P}_y[y, y'] \oplus p(x', y'), & \text{if } x \neq y. \end{cases} \tag{9}$$

Next we use induction on the length of $xy$ to prove the claim of the lemma. The base case corresponds to $x = y$. In this case $p(x, y)$ degenerates to a point and $|p(x, y)| = 0$. To prove the inductive step, pick a shortest side $xy$ of a quadrilateral $acbd$, with $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4$ crossing each other, and assume that the lemma holds for all such sides shorter than $xy$. Let $p(x, y)$ be the path determined recursively as in (9). By the inductive hypothesis, we have that $p(x', y')$ contains no edges longer than $|x'y'| \leq |xy|$, and

$$|p(x', y')| \leq \frac{6}{\sqrt{2}-1}|x'y'| \leq \frac{6}{2-\sqrt{2}}|xy|. \tag{10}$$

This latter inequality follows from (8). Also recall that no edge on $\mathcal{P}_x$ and $\mathcal{P}_y$ is longer than $|xy|$, which together with formula (9) and the arguments above, implies that no edge on $p(x, y)$ is longer than $|xy|$. Substituting inequalities 10 and $(ii)$ from Lemma 7 in formula (9) yields

$$|p(x,y)| \leq (3\sqrt{2} + \frac{6}{2-\sqrt{2}}) \cdot |xy| = \frac{6}{\sqrt{2}-1} \cdot |xy|.$$

This completes the proof. □

## 4. $Y_4^\infty$ and $Y_4$

The final step of our analysis is to prove that every individual edge of $Y_4^\infty$ is spanned by a short path in $Y_4$. This, along with the result of Theorem 1, establishes that $Y_4$ is a spanner.

Fix an edge $\overrightarrow{ab} \in Y_4^\infty$. Call an edge or a path *t-short* (with respect to $|ab|$) if its length is within a constant factor $t$ of $|ab|$. In our proof that $ab$ is spanned by a $t$-short path in $Y_4$, we will make use of the following three statements (proved in the Appendix).

**S1** If $xy$ is $t$-short, then $\mathcal{P}_R(x \to y)$, and therefore its reverse, $\mathcal{P}_R^{-1}(x \to y)$ are $t\sqrt{2}$-short by Lemma 2.

**S2** If $xy \in Y_4$ is $t_1$-short and $zw \in Y_4$ is $t_2$-short, and if $xy$ intersects $zw$, Lemma 4(ii) and Lemma 8 show that there is a $t_3$-short path between any two of the endpoints of these edges, with $t_3 = t_1 + t_2 + 3(2+\sqrt{2})\max(t_1, t_2)$.

**S3** If $p(x, y)$ is a $t_1$-short path and $p(z, w)$ is a $t_2$-short path and these two paths intersect, then by **S2** there is a $t_3$-short path $P$ between any two of the endpoints of these paths, with $t_3 = t_1 + t_2 + 3(2+\sqrt{2})\max(t_1, t_2)$.

**Lemma 9.** *Fix an edge $ab \in Y_4^\infty$. There is a path $p(a, b) \in Y_4$ between $a$ and $b$, of length $|p(a, b)| \leq t|ab|$, for $t = 26 + 23\sqrt{2}$.*

**Proof.** For the sake of clarity, we only prove here that there is a short path $p(a, b)$ between $a$ and $b$, and and defer the calculation of the actual stretch factor $t$ to the Appendix. We refer to an edge or a path as *short* if its length is within a constant factor of $|ab|$. Assume without loss of generality that $\overrightarrow{ab} \in Q_1(a)$. If $\overrightarrow{ab} \in Y_4$, then $p(a, b) = ab$ and the proof is finished. So assume the opposite, and let $\overrightarrow{ac}$ be the edge in $Y_4$ that lies in $Q_1(a)$; since $Q_1(a)$ is nonempty, $\overrightarrow{ac}$ exists. Because $\overrightarrow{ac} \in Y_4$ and $b$ is in the same quadrant of $a$ as $c$, we have that

$$\begin{aligned} |ac| &\leq |ab| && \text{(i)} \\ |bc| &< |ac|\sqrt{2} && \text{(ii).} \end{aligned} \tag{11}$$

Inequality (ii) above follows immediately from the Law of Cosines, which implies that $|bc|^2 < |ab|^2 + |ac|^2$ (because the angle formed by $ab$ and $ac$ is strictly less than $\pi/2$), and the fact that $|ac| \leq |ab|$. Thus both $ac$ and $bc$ are short. And this in turn implies that $\mathcal{P}_R(b \to c)$ is short by **S1**. We next focus on $\mathcal{P}_R(b \to c)$. For simplicity, we assume that $ac$ is counterclockwise of $ab$; the situation when $ac$ lies clockwise of $ab$ is symmetrical. Let $b' \notin R(b, c)$ be the other endpoint of $\mathcal{P}_R(b \to c)$. We distinguish three cases.

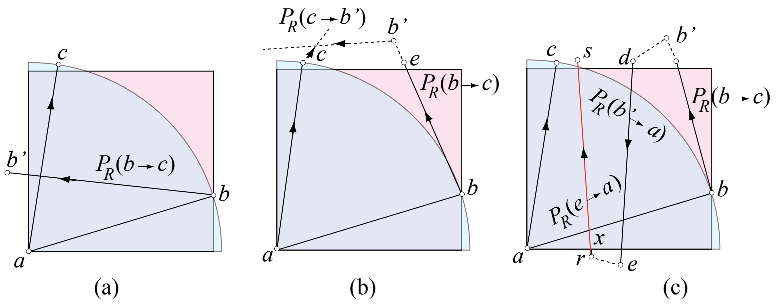


Fig. 6. Lemma 9: (a) Case 1: $\mathcal{P}_R(b \to c)$ and $ac$ have a nonempty intersection. (b) Case 2: $\mathcal{P}_R(b' \to a)$ and $ab$ have an empty intersection. (c) Case 3: $\mathcal{P}_R(b' \to a)$ and $ab$ have a non-empty intersection.

**Case 1:** $\mathcal{P}_R(b \to c)$ and $ac$ intersect (see Fig. 6a). Then by **S3** there is a short path $p(a, b)$ between $a$ and $b$.

**Case 2:** $\mathcal{P}_R(b \to c)$ and $ac$ do not intersect, and $\mathcal{P}_R(b' \to a)$ and $ab$ do not intersect (see Fig. 6b). Note that because $b'$ is the endpoint of the short path $\mathcal{P}_R(b \to c)$, the triangle inequality on $\triangle abb'$ implies that $ab'$ is short, and therefore $\mathcal{P}_R(b' \to a)$ is short, by **S1**. We consider two cases:

(i) $\mathcal{P}_R(b' \to a)$ intersects $ac$. Then by **S3** there is a short path $p(a, b')$. So

$$p(a, b) = p(a, b') \oplus \mathcal{P}_R^{-1}(b \to c)$$

is short.

(ii) $\mathcal{P}_R(b' \to a)$ does not intersect $ac$. Then $\mathcal{P}_R(c \to b')$ must intersect $\mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$. Next we establish that $b'c$ is short. Let $\overrightarrow{eb'}$ be the last edge of $\mathcal{P}_R(b \to c)$, and so incident to $b'$ (note that $e$ and $b$ may coincide). Because $\mathcal{P}_R(b \to c)$ does not intersect $ac$, $b'$ and $c$ are in the same quadrant for $e$. It follows that $|eb'| \le |ec|$ and $\angle b'ec < \pi/2$. These observations along with Proposition 2 for $\triangle b'ec$ imply that $|b'c|^2 < |b'e|^2 + |ec|^2 \le 2|ec|^2 < 2|bc|^2$ (this latter inequality uses the fact that $\angle bec > \pi/2$, which implies that $|ec| < |bc|$). It follows that

$$|b'c| \le |bc|\sqrt{2} \ \le \ 2|ac| \qquad \text{(by (11)ii).} \qquad (12)$$

Thus $b'c$ is short, and by **S1** we have that $\mathcal{P}_R(c \to b')$ is short. Since $\mathcal{P}_R(c \to b')$ intersects the short path $\mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$, there is by **S3** a short path

$p(c, b)$, and so

$$p(a, b) = ac \oplus p(c, b)$$

is short.

**Case 3:** $\mathcal{P}_R(b \to c)$ and $ac$ do not intersect, and $\mathcal{P}_R(b' \to a)$ intersects $ab$ (see Fig. 6c). If $\mathcal{P}_R(b' \to a)$ intersects $ab$ at $a$, then $p(a, b) = \mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$ is short. So assume otherwise, in which case there is an edge $\overrightarrow{de} \in \mathcal{P}_R(b' \to a)$ that crosses $ab$. Then $d \in Q_1(a)$, $e \in Q_3(a) \cup Q_4(a)$, and $e$ and $a$ are in the same quadrant for $d$. Note however that $e$ cannot lie in $Q_3(a)$, since in that case $\angle dae > \pi/2$, which would imply $|de| > |da|$, which in turn would imply $\overrightarrow{de} \notin Y_4$. So it must be that $e \in Q_4(a)$.

Next we show that $\mathcal{P}_R(e \to a)$ does not cross $ab$. Assume the opposite, and let $\overrightarrow{rs} \in \mathcal{P}_R(e \to a)$ cross $ab$. Then $r \in Q_4(a)$, $s \in Q_1(a) \cup Q_2(a)$, and $s$ and $a$ are in the same quadrant for $r$. Arguments similar to the ones above show that $s \notin Q_2(a)$, so $s$ must lie in $Q_1(a)$. Let $\delta$ be the $L_\infty$ distance from $a$ to $b$. Let $x$ be the projection of $r$ on the horizontal line through $a$. Then

$$|rs| \geq |rx| + \delta \geq |rx| + |xa| > |ra| \quad \text{(by the triangle inequality)}$$

Because $a$ and $s$ are in the same quadrant for $r$, the inequality above contradicts $\overrightarrow{rs} \in Y_4$.

We have established that $\mathcal{P}_R(e \to a)$ does not cross $ab$. Then $\mathcal{P}_R(a \to e)$ must intersect $\mathcal{P}' = de \oplus \mathcal{P}_R(e \to a)$. Note that $de$ is short because it is in the short path $\mathcal{P}_R(b' \to a)$. Thus $ae$ is short (because $|ae| < |ai| + |ei| < |ab| + |ed|$, where $i$ is the intersection point between $ab$ and $de$), and so $\mathcal{P}_R(a \to e)$ and $\mathcal{P}_R(e \to a)$ are short, by **S1**. Then the short path $\mathcal{P}_R(a \to e)$ intersects either $de$ or $\mathcal{P}_R(e \to a)$, each of which is short, and by **S3** there is a short path $p(a, e)$. Then

$$p(a, b) = p(a, e) \oplus \mathcal{P}_R^{-1}(b' \to a) \oplus \mathcal{P}_R^{-1}(b \to c)$$

is short. Straightforward calculations detailed in the appendix show that, in each of these cases, the stretch factor for $p(a, b)$ does not exceed $26 + 23\sqrt{2}$. □

Our main result follows immediately from Theorem 1 and Lemma 9:

**Theorem 2.** *$Y_4$ is a t-spanner, for $t \geq 8\sqrt{2}(26 + 23\sqrt{2})$.*

## 5. Conclusion

Our results settle a long-standing open problem, asking whether $Y_4$ is a spanner or not. We answer this question positively, and establish a loose stretch factor of $8\sqrt{2}(26 + 23\sqrt{2})$. Finding tighter stretch factors for both $Y_4^\infty$ and $Y_4$ remain interesting open problems. Establishing whether or not $Y_5$ is a spanner is also open.

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## References

1. P. Bose, M. Damian, K. Douïeb, J. O'Rourke, B. Seamone, M. Smid and S. Wuhrer, $\pi/2$-Angle Yao Graphs are Spanners, *Technical Report* (2010) arXiv:1001.2913v1.
2. P. Bose, A. Maheshwari, G. Narasimhan, M. Smid and N. Zeh, Approximating geometric bottleneck shortest paths, *Computational Geometry: Theory and Applications* (2004) 29:233–249.
3. M. Damian, N. Molla and V. Pinciu, Spanner properties of $\pi/2$-angle Yao graphs, in *Proc. of the 25th European Workshop on Computational Geometry* (March 2009) pp. 21–24.
4. M. Damian and K. Raudonis, Yao graphs span Theta graphs, in *Proc. of the 4th Annual International Conference on Combinatorial Optimization and Applications* (December 2010) pp. 181–194.
5. M. Fischer, T. Lukovszki and M. Ziegler, Geometric searching in walkthrough animations with weak spanners in real time, in *Proc. of the 6th Annual European Symposium on Algorithms* (1998) pp. 163–174.
6. N. Molla, Yao spanners for wireless ad hoc networks, *M.S. Thesis*, Department of Computer Science, Villanova University (December 2009).
7. J.W. Green, A note on the chords of a convex curve, *Portugaliae Mathematica* (1951) 10(3):121–123.
8. A.C.-C. Yao, On constructing minimum spanning trees in $k$-dimensional spaces and related problems, *SIAM Journal on Computing* (1982) 11(4):721–736.

## 6. Appendix

### 6.1. *Proof of Lemma 4*

For any node $a \in V$, let $D(a, r)$ denote the open disk centered at $a$ of radius $r$, and let $\partial D(a, r)$ denote the boundary of $D(a, r)$.

**Proof.** The first part of the lemma is a well-known fact that holds for any quadrilateral (see Ref. [7], for instance). For the second part of the lemma, let $ab$ be the shorter of the diagonals of $acbd$, and assume without loss of generality that $\overrightarrow{ab} \in Q_1(a)$. Imagine two disks $D_a = D(a, |ab|)$ and $D_b = D(b, |ab|)$, as in Fig. 7a. If either $c$ or $d$ belongs to $D_a \cup D_b$, then the lemma follows: a shortest quadrilateral edge is shorter than $|ab|$.

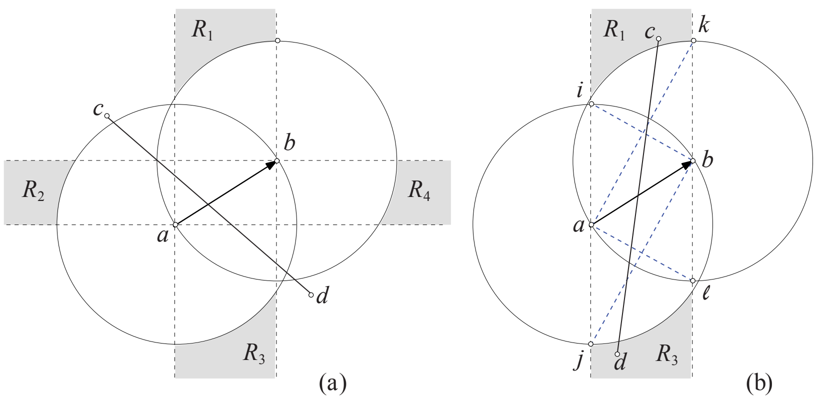


Fig. 7. Lemma 4 (a) $c \notin R_1 \cup R_2 \cup R_3 \cup R_4$ (b) $c \in R_1$.

So suppose that neither $c$ nor $d$ lies in $D_a \cup D_b$. In this case, we use the fact that $cd$ crosses $ab$ to show that $\overrightarrow{cd}$ cannot be an edge in $Y_4$. Define the following regions (see Fig. 7a):

$$
\begin{aligned}
R_1 &= (Q_1(a) \cap Q_2(b)) \backslash (D_a \cup D_b) \\
R_2 &= (Q_2(a) \cap Q_3(b)) \backslash (D_a \cup D_b) \\
R_3 &= (Q_4(a) \cap Q_3(b)) \backslash (D_a \cup D_b) \\
R_4 &= (Q_1(a) \cap Q_4(b)) \backslash (D_a \cup D_b).
\end{aligned}
$$

If the node $c$ is not inside any of the regions $R_i$, for $i = \{1, 2, 3, 4\}$, then the nodes $a$ and $b$ are in the same quadrant of $c$ as $d$. In this case, note that either $\angle cad > \pi/2$ or $\angle cbd > \pi/2$, which implies that either $|ca|$ or $|cb|$ is strictly smaller than $|cd|$. These together show that $\overrightarrow{cd} \notin Y_4$.

So assume that $c$ is in $R_i$ for some $i \in \{1, 2, 3, 4\}$. In this situation, the node $d$ must lie in the region $R_j$, with $j = (i + 2) \bmod 4$ (with the understanding that

$R_0 = R_4$), because otherwise, either (i) $a$ and $d$ are in the same quadrant of $c$ and $|ca| < |cd|$ or (ii) $b$ and $d$ are in the same quadrant of $c$ and $|cb| < |cd|$. Either case contradicts the fact $\overrightarrow{cd} \in Y_4$. Consider now the case $c \in R_1$ and $d \in R_3$; the other cases are treated similarly. Let $i$ and $j$ be the intersection points between $D_a$ and the vertical line through $a$. Similarly, let $k$ and $\ell$ be the intersection points between $D_b$ and the vertical line through $b$ (see Fig. 7b). Since $ij$ is a diameter of $D_a$, we have that $\angle ibj = \pi/2$ and similarly $\angle kal = \pi/2$. Also note that $\angle cbd \geq \angle ibj = \pi/2$, meaning that $|cd| > |cb|$. Similarly, $\angle cad \geq \angle kal = \pi/2$, meaning that $|cd| > |ca|$. These along with the fact that at least one of $a$ and $b$ is in the same quadrant for $c$ as $d$, imply that $\overrightarrow{cd} \notin Y_4$. This completes the proof. □

### 6.2. *Proof of Lemma 5*

**Proof.** The proof consists of two parts showing that the following claims hold: (I) $d \in Q_2(a)$ and (II) $\mathcal{P}_R(d \to a)$ does not cross $ab$. Before we prove these two claims, let us argue that they are sufficient to prove the lemma. Lemma 3 and claim (I) imply that $\mathcal{P}_R(a \to d)$ cannot cross $cd$, because $\mathcal{P}_R(a \to d) \in Q_2(a)$ and $cd \in Q_2(c)$ are in quadrants of identical indices. As a result, $\mathcal{P}_R(a \to d)$ intersects the left side of the rectangle $R(d, a)$. Consider the last edge $\overrightarrow{xy}$ of the path $\mathcal{P}_R(d \to a)$. If this edge crosses the right side of $R(a, d)$, then claim (II) implies that $y$ is in the wedge bounded by $ab$ and the upwards vertical ray starting at $a$; this further implies that $|ay| < |ab|$, contradicting the fact that $\overrightarrow{ab}$ is an edge in $Y_4$. Therefore, $\overrightarrow{xy}$ intersects the bottom side of $R(d, a)$, and the lemma follows (see Fig. 8b).

To prove the first claim (I), we observe that the lemma assumptions imply that $d \in Q_1(a) \cup Q_2(a)$. Therefore, it suffices to prove that $d$ is not in $Q_1(a)$. Assume to the contrary that $d \in Q_1(a)$. Since $c \in Q_1(a)$, it must be that $b \in Q_2(c)$; otherwise, $\angle acb \geq \pi/2$, which implies $|ab| > |ac|$, contradicting the fact that $\overrightarrow{ab} \in Y_4$. Let $i$ and $j$ be the intersection points between $cd$ and $\partial D(a, |ab|)$, where $i$ is to the left of $j$. Since $\angle dbc \geq \angle ibj > \pi/2$, we have $|cb| < |cd|$. This, together with the fact that $b$ and $d$ are in the same quadrant $Q_2(c)$, contradicts the assumption that $\overrightarrow{cd}$ is an edge in $Y_4$. This completes the proof of claim (I).

Next we prove claim (II) by contradiction. Thus, we assume that there is an edge $\overrightarrow{xy}$ on the path $\mathcal{P}_R(d \to a)$ that crosses $ab$. Then necessarily $x \in R(a, d)$ and $y \in Q_1(a) \cup Q_4(a)$. If $y \in Q_4(a)$, then $\angle xay > \pi/2$, meaning that $|xy| > |xa|$, a contradiction to the fact that $\overrightarrow{xy} \in Y_4$. Thus, it must be that $y \in Q_1(a)$, as in Fig. 8a. This implies that $|ab| \leq |ay|$, because $\overrightarrow{ab} \in Y_4$.

The contradiction to our assumption that $\overrightarrow{xy}$ crosses $ab$ will be obtained by proving that $|xy| > |xa|$. Indeed, this inequality contradicts the fact that $\overrightarrow{xy} \in Y_4$, because both $a$ and $y$ are in $Q_4(x)$, and $Y_4$ would have picked $\overrightarrow{xa}$ in place of $\overrightarrow{xy}$.

Let $\delta$ be the distance from $x$ to the horizontal line through $a$. Our intermediate goal is to show that

$$\delta \leq |ab|/\sqrt{2}. \tag{13}$$

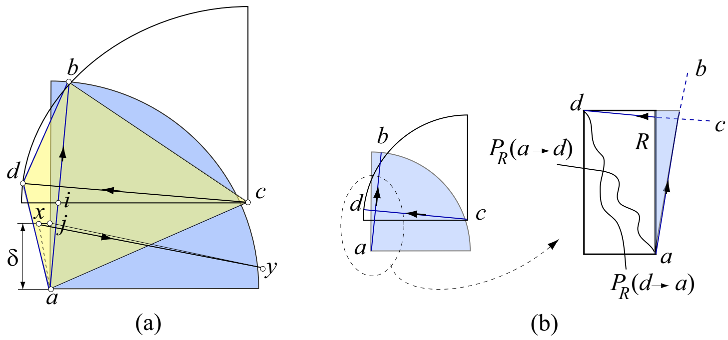


Fig. 8. (a) Lemma 5: $xy \in \mathcal{P}_R(d \to a)$ cannot cross $ab$.

We claim that $\angle acb < \pi/2$. Indeed, if this is not the case, then $|ac| < |ab|$, contradicting the fact that $\overrightarrow{ab}$ is an edge in $Y_4$. By a similar argument, and using the fact that $\overrightarrow{cd}$ is an edge in $Y_4$, we obtain the inequality $\angle cbd < \pi/2$. We now consider two cases, depending on the relative lengths of $ac$ and $cb$.

(1) Assume first that $|ac| > |cb|$. If $\angle cad \geq \pi/2$, then $|cd| \geq |ac| > |cb|$, contradicting the fact that $\overrightarrow{cd}$ is an edge in $Y_4$ (recall that $b$ and $d$ are in the same quadrant of $c$). Therefore, we have $\angle cad < \pi/2$. So far we have established that three angles of the convex quadrilateral $acbd$ are acute. It follows that the fourth one ($\angle adb$) is obtuse. Proposition 2 applied to $\triangle adb$ tells us that

$$|ab|^2 > |ad|^2 + |db|^2 \geq 2|ad|^2,$$

where the latter inequality follows from the assumption that $ad$ is a shortest side of $acbd$ (and, therefore, $|db| \geq |ad|$). Thus, we have that $|ad| \leq |ab|/\sqrt{2}$. This along with the fact that $x \in R(a,d)$ implies inequality (13).

(2) Assume now that $|ac| \leq |cb|$. Let $i$ be the intersection point between $ab$ and the horizontal line through $c$ (refer to Fig. 8a). Note that $\angle aic \geq \pi/2$ and $\angle bic \leq \pi/2$ (these two angles sum to $\pi$). This along with Proposition 2 applied to triangle $\triangle aic$ shows that

$$|ac|^2 \geq |ai|^2 + |ic|^2.$$

Similarly, Proposition 2 applied to triangle $\triangle bic$ shows that

$$|bc|^2 \leq |bi|^2 + |ic|^2.$$

The two inequalities above along with our assumption that $|ac| \leq |cb|$ imply that $|ai| \leq |bi|$, which in turn implies that $|ai| \leq |ab|/2$, because $|ai| + |ib| = |ab|$. Since $x$ is below $i$ (otherwise, $|cx| < |cd|$, contradicting the fact that $\overrightarrow{cd}$ is an edge in $Y_4$), we have $\delta \leq |ai|$. It follows that $\delta \leq |ab|/2$.

Finally we derive a contradiction using the now established inequality (13). Let $j$ be the orthogonal projection of $x$ onto the vertical line through $a$ (thus $|aj| = \delta$).

Note that $\angle ajy < \pi/2$, because $y \in Q_4(x)$. By Proposition 2 applied to $\triangle ajy$, we have

$$|ay|^2 < |aj|^2 + |jy|^2 = \delta^2 + |jy|^2.$$

Since $y$ and $b$ are in the same quadrant of $a$, and since $\overrightarrow{ab} \in Y_4$, we have that $|ab| \leq |ay|$. This along with the inequality above and (13) implies that $|jy| \geq |ab|/\sqrt{2} \geq \delta$. By Proposition 2 applied to $\triangle xjy$, we have $|xy|^2 > |xj|^2 + |jy|^2 \geq |xj|^2 + \delta^2 = |xj|^2 + |ja|^2 = |xa|^2$. It follows that $|xy| > |xa|$, contradicting our assumption that $\overrightarrow{xy} \in Y_4$. □

### 6.3. *Proof of Lemma 6*

**Proof.** We first show that $d \notin Q_3(a)$. Assume the opposite. Since $c \in Q_1(a)$ and

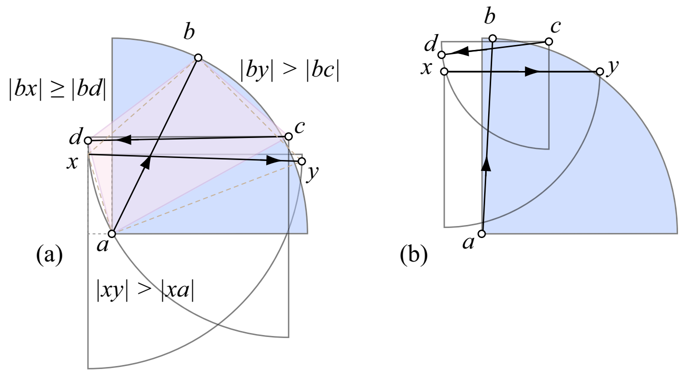


Fig. 9. Lemma 6: (a) $\mathcal{P}_R(d \to a)$ does not cross $ab$. (b) If $ad$ is not the shortest side of $acbd$, the lemma conclusion might not hold.

$d \in Q_3(a)$, we have that $\angle cad > \pi/2$. This implies that $|ca| < |cd|$, which along with the fact that $a, d \in Q_3(c)$ contradict the fact that $\overrightarrow{cd} \in Y_4$. Also note that $d \notin Q_1(a)$, since in that case $ab$ and $cd$ could not intersect. In the following we discuss the case $d \in Q_2(a)$; the case $d \in Q_4(a)$ is symmetric.

A first observation is that $c$ must lie below $b$; otherwise $|cb| < |cd|$ (since $\angle cbd > \pi/2$), which would contradict the fact that $\overrightarrow{cd} \in Y_4$. We now prove by contradiction that there is no edge in $\mathcal{P}_R(d \to a)$ crossing $ab$. Assume the contrary, and let $\overrightarrow{xy} \in \mathcal{P}_R(d \to a)$ be such an edge. Then necessarily $x \in R(a, d)$ and $\overrightarrow{xy} \in Q_4(x)$. Note that $y$ cannot lie below $a$; otherwise $|xa| < |xy|$ (since $\angle xay > \pi/2$), which would contradict the fact that $\overrightarrow{xy} \in Y_4$. Also $y$ must lie outside $D(c, |cd|) \cap Q(c, d)$, otherwise $\overrightarrow{cd}$ could not be in $Y_4$. These together show that $y$ sits to the right of $c$. See Fig. 9a. Then the following inequalities regarding the quadrilateral $xayb$ must hold:

(i) $|by| > |bc|$, due to the fact that $\angle bcy > \pi/2$.
(ii) $|bx| \geq |bd|$ ($|bx| = |bd|$ if $x$ and $d$ coincide). If $x$ and $d$ are distinct, the inequality $|bx| > |bd|$ follows from the fact that $|cx| \geq |cd|$ (since $x$ is outside $D(c, |cd|)$), and Proposition 1 applied to the quadrilateral $xcbd$:

$$|bd| + |cx| < |bx| + |cd|$$

Inequalities (i) and (ii) show that $by$ and $bx$ are longer than sides of the quadrilateral $acbd$, and so they must be longer than the shortest side of $acbd$, which by assumption (ii) of the lemma is $ad$: $\min\{|bx|, |by|\} \geq |ad| \geq |ax|$ (this latter inequality follows from the fact that $x \in R(d, a)$). Also note that $|ab| \leq |ay|$, since $\overrightarrow{ab} \in Y_4$ and $y$ lies in the same quadrant of $a$ as $b$. The fact that both diagonals of $xayb$ are in $Y_4$ enables us to apply Lemma 4(ii) to conclude that $ay$ is not a shortest side of the quadrilateral $xayb$. Thus $xa$ is a shortest side of the quadrilateral $xayb$, and we can use Lemma 4(ii) to claim that

$$|xa| < \min\{|xy|, |ab|\} \leq |xy|.$$

This contradicts our assumption that $\overrightarrow{xy} \in Y_4$. □

Fig. 9(b) shows that the claim of the lemma might be false without assumption (ii).

### 6.4. *Calculations for the stretch factor of $p(a, b)$ in Lemma 9*

We start by computing the stretch factor of the short paths claimed by statements **S2** and **S3**.

**S2** If $xy \in Y_4$ and $zw \in Y_4$ are short, and if $xy$ intersects $zw$, then there is a short path $P$ between any two of the endpoints of these edges, of length

$$|P| \leq |xy| + |zw| + 3(2 + \sqrt{2}) \max\{|xy|, |zw|\}. \tag{14}$$

This upper bound can be derived as follows. Let $ij$ be a shortest side of the quadrilateral $xzyw$. By Lemma 8, $Y_4$ contains a path $p(i, j)$ no longer than $6(\sqrt{2}+1)|ij|$. By Lemma 4, $|ij| \leq \max\{|xy|, |zw|\}/\sqrt{2}$. These together with the fact that $|P| \leq |xy| + |zw| + |p(i, j)|$ yield inequality (14).

**S3** Here we prove a tighter version of this statement: If $p(x, y)$ and $p(z, w)$ are short paths that intersect, then there is a short path $P$ between any two of the endpoints of these paths, of length

$$|P| \leq |p(x, y)| + |p(z, w)| + 3(2 + \sqrt{2}) \max\{|xy|, |zw|\}. \tag{15}$$

This follows immediately from **S2** and the fact that no edge of $p(x, y) \cup p(z, w)$ is longer than $\max\{|xy|, |zw|\}$ (by Lemma 8).

**Case 1:** $\mathcal{P}_R(b \to c)$ and $ac$ intersect. Then by **S3** we have

$$
\begin{aligned}
|p(a,b)| &\le |\mathcal{P}_R(b,c)| + |ac| + 3(2+\sqrt{2})\max\{|bc|,|ac|\} \\
&\le \sqrt{2}|bc| + |ac| + 3(2+\sqrt{2})\sqrt{2}|ac| && \text{(by (7), (11)ii)} \\
&= 3(3+2\sqrt{2})|ac| \le 3(3+2\sqrt{2})|ab| && \text{(by (11)i).}
\end{aligned}
$$

**Case 2(i):** $\mathcal{P}_R(b \to c)$ and $ac$ do not intersect; $\mathcal{P}_R(b' \to a)$ and $ab$ do not intersect; and $\mathcal{P}_R(b' \to a)$ intersects $ac$. By **S3**, there is a short path $p(a,b')$ of length

$$
\begin{aligned}
|p(a,b')| &\le |\mathcal{P}_R(b',a)| + |ac| + 3(2+\sqrt{2})\max\{|b'a|,|ac|\} \\
&\le |b'a|\sqrt{2} + |ac| + 3(2+\sqrt{2})\max\{|b'a|,|ac|\} && \text{(by (7)).} \qquad (16)
\end{aligned}
$$

Next we establish an upper bound on $|b'a|$. By the triangle inequality,

$$|ab'| < |ac| + |cb'| \;\le\; 3|ac| \qquad \text{(by (12)).} \qquad (17)$$

Substituting this inequality in (16) yields

$$|p(a,b')| \le (19+12\sqrt{2})|ac|. \qquad (18)$$

Thus $p(a,b) = p(a,b') \oplus \mathcal{P}_R^{-1}(b \to c)$ is a path in $Y_4$ of length

$$
\begin{aligned}
|p(a,b)| &\le |p(a,b')| + |bc|\sqrt{2} && \text{(by (7))} \\
&\le |p(a,b')| + 2|ac| && \text{(by (11)ii)} \\
&\le (21+12\sqrt{2})|ac| && \text{(by (18))} \\
&\le (21+12\sqrt{2})|ab| && \text{(by (11)i).}
\end{aligned}
$$

**Case 2(ii):** $\mathcal{P}_R(b \to c)$ and $ac$ do not intersect; $\mathcal{P}_R(b' \to a)$ and $ab$ do not intersect; and $\mathcal{P}_R(b' \to a)$ does not intersect $ac$. Then $\mathcal{P}_R(c \to b')$ must intersect $\mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$. By **S3** there is a short path $p(c,b)$ of length

$$
\begin{aligned}
|p(c,b)| &\le |\mathcal{P}_R(c \to b')| + |\mathcal{P}_R(b \to c)| + |\mathcal{P}_R(b' \to a)| + 3(2+\sqrt{2})\max\{|cb'|,|bc|,|b'a|\} \\
&\le (|cb'| + |bc| + |b'a|)\sqrt{2} + 3(2+\sqrt{2})\max\{|cb'|,|bc|,|b'a|\} && \text{(by (7)).}
\end{aligned}
$$

Inequalities (11)ii, (12) and (17) imply that $\max\{|cb'|,|bc|,|b'a|\} \le 3ac$. Substituting in the above, we get

$$
\begin{aligned}
|p(c,b)| &\le (2+\sqrt{2}+3)\sqrt{2}|ac| + 9(2+\sqrt{2})|ac| \\
&\le (20+14\sqrt{2})|ac| && \text{(by (11)i).}
\end{aligned}
$$

Thus $p(a,b) = ac \oplus p(c,b)$ is a path in $Y_4$ from $a$ to $b$ of length

$$|p(a,b)| \le (21+14\sqrt{2})|ac| \;\le\; (21+14\sqrt{2})|ab| \qquad \text{(by (11)i).}$$

**Case 3:** $\mathcal{P}_R(b \to c)$ and $ac$ do not intersect, and $\mathcal{P}_R(b' \to a)$ intersects $ab$. If $\mathcal{P}_R(b' \to a)$ intersects $ab$ at $a$, then $p(a,b) = \mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$ is clearly short and does not exceed the spanning ratio of the lemma. Otherwise, there is an edge $\overrightarrow{de} \in \mathcal{P}_R(b' \to a)$ that crosses $ab$, and $\mathcal{P}_R(a \to e)$ intersects $de \oplus \mathcal{P}_R(e \to a)$ (as established in the proof of Lemma 9). If $\mathcal{P}_R(a \to e)$ intersects $de$, then by **S3** there is a short path $p(a,e)$ of length

$$|p(a,e)| \leq |\mathcal{P}_R(a \to e)| + |de| + 3(2+\sqrt{2}) \max\{|ae|, |de|\} \tag{19}$$

Otherwise, if $\mathcal{P}_R(a \to e)$ intersects $\mathcal{P}_R(e \to a)$, then by **S3** there is a short path $p(a,e)$ of length

$$|p(a,e)| \leq |\mathcal{P}_R(a \to e)| + |\mathcal{P}_R(e \to a)| + 3(2+\sqrt{2})|ae| \tag{20}$$

A loose upper bound on $|ae|$ can be obtained by employing Proposition 1 to the quadrilateral $aebd$: $|ae| + |bd| < |ab| + |de| < |ab| + |ab'|$. Substituting the upper bound for $ab'$ from (17) yields

$$|ae| < |ab| + 3|ac| \leq 4|ab|. \tag{21}$$

By Lemma 2, $|de| \leq |ab'|$ (since $de \in \mathcal{P}_R(b' \to a)$), which along with (17) implies

$$|de| \leq 3|ab|. \tag{22}$$

Substituting inequalities (7), (21) and (22) in (19) yields

$$|p(a,e)| \leq (27 + 16\sqrt{2})|ab|.$$

Substituting inequalities (7) and (21) in (20) gives

$$|p(a,e)| \leq (24 + 20\sqrt{2})|ab|,$$

which is a looser upper bound that applies to both cases. Then

$$p(a,b) = p(a,e) \oplus \mathcal{P}_R^{-1}(b' \to a) \oplus \mathcal{P}_R^{-1}(b \to c)$$

is a path from $a$ to $b$ of length

$$\begin{aligned} |p(a,b)| &\leq (24 + 20\sqrt{2})|ab| + 3\sqrt{2}|ab| + 2|ab| \qquad \text{(by (23), (17), (11))} \\ &= (26 + 23\sqrt{2})|ab|. \end{aligned}$$