Some Properties of Yao Y_4 Subgraphs

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Abstract

The Yao graph for $k = 4$, Y_4, is naturally partitioned into four subgraphs, one per quadrant. We show that the subgraphs for one quadrant differ from the subgraphs for two adjacent quadrants in three properties: planarity, connectedness, and whether the directed graphs are spanners.

1 Introduction

The Yao graph is defined for an integer parameter k; here we study only $k = 4$, and call $\overrightarrow{Y_4}$ the directed Yao graph, and Y_4 the undirected version. For a set of points P, $\overrightarrow{Y_4}$ connects each point to its closest neighbor in each of the four quadrants surrounding it, defined as in Figure 1. Ties are broken arbitrarily. The undirected graph Y_4 simply ignores the direction.

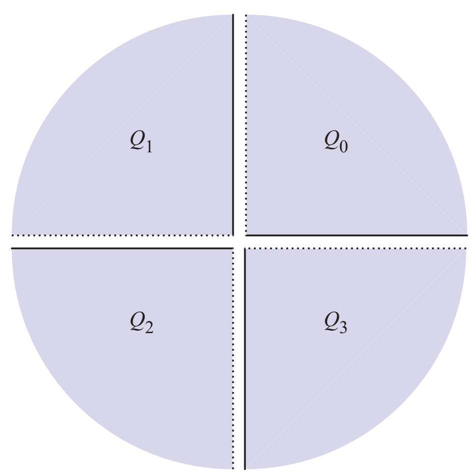

Figure 1: Definition of quadrants. Solid lines are closed, dotted lines are open.

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The question of whether Y_4 is a spanner was raised in [DMP09]. A t-spanner has the property that the path between a and b in the graph is no longer than $t|ab|$, for a constant t. In this note, we do not further motivate the study of Y_4, but rather investigate some properties of subgraphs of Y_4, which may ultimately have some bearing on whether it is a spanner.

We make two "general position" assumptions:

1. No two pair of points determine the same distance (so there are no ties).

2. No two points share a vertical or horizontal coordinate.

These assumptions simplify the presentation. In this note, we will not explore whether the assumptions can be removed while retaining all the results.

Notation. $Q_i(a, b)$ is the circular quadrant whose origin is at a and which reaches out to b. Often the subscript i will be dropped, as it is determined by a and b. $Q_i(a)$ is the unbounded quadrant with corner at a. Thus, $Q_i(a, b) = Q_i(a) \cap \text{disk}(a, |ab|)$. $R(a, b)$ is the closed rectangle with opposite corners a and b.

We focus on two adjacent quadrants, Q_0 and Q_1. Let $Y_4^{\{\lambda\}}$ be the Y_4 graph restricted to the quadrants in the list λ. See Figure 2 for examples.

Our results are summarized in Table 1.

Property	$Y_4^{\{i\}}$	$Y_4^{\{i,i+1\}}$
Planarity	planar	not planar
Connectedness	not connected	connected
Undirected spanner	not a spanner	not a spanner
Directed spanner	spanner	not a spanner

Table 1: Summary of Results

2 Planarity

It is known that $Y_4^{\{i\}}$ is a planar forest, in general disconnected; see Figure 2(a,b). This is folklore,[1] but we offer a proof of planarity.

Lemma 1 *No two edges of $Y_4^{\{i\}}$ properly cross.*

Proof: Let both ab and cd be in $Y_4^{\{0\}}$, and suppose ab and cd properly cross. see Figure 3. The quadrants $Q(a, b)$ and $Q(c, d)$ must be empty of points. We consider three cases, depending on the location of c w.r.t. a.

[1] Mirela Damian [private communication, Feb. 2009].

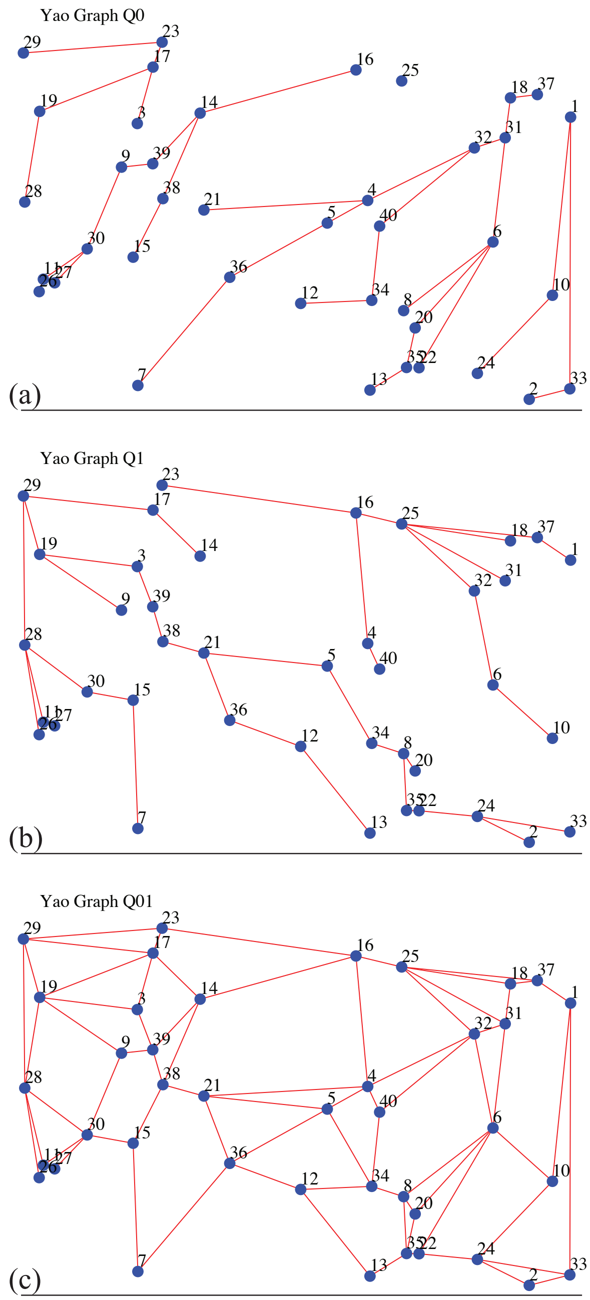
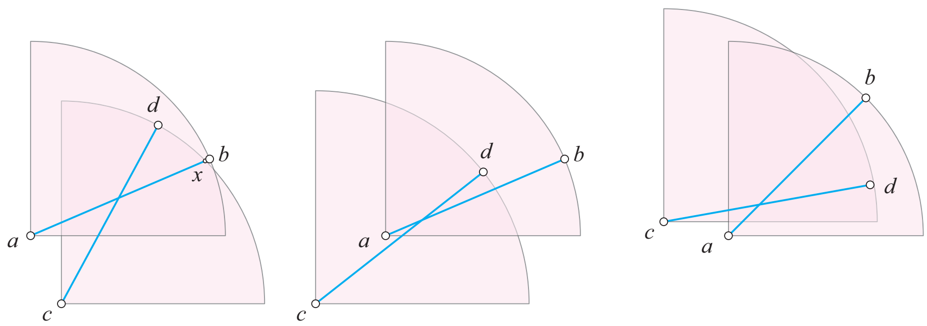


Figure 2: $Y_4^{\{0\}}$, $Y_4^{\{1\}}$, and $Y_4^{\{0,1\}}$, for the same 40-point set.



(a) (b) (c)

Figure 3: ab and cd may not cross.

1. $c \in Q_3(a)$. Then cd crosses ab from below. We analyze just this case in detail. Because $b \notin Q(c, d)$, the circular boundary of $Q(c, d)$ must cut ab, say at x. Consider two further cases
   (a) The slope of the arc of $Q(c, d)$ at x is shallower than the slope of the arc of $Q(a, b)$ at b; see Figure 3(a). Then $d \in Q(a, b)$.
   (b) The slope at x is equal to or steeper than that at b. Then, because c is strictly below a, the radius $|cd|$ is greater than $|ab|$. But then c cannot be in $Q_3(a)$.
2. $c \in Q_2(a)$. Then cd could cross ab from below, Figure 3(b), or from above, Figure 3(c). In both cases, a quadrant that must be empty is not.
3. $c \in Q_1(a)$. This case is the same as the first case, with the roles of a and c interchanged.

□

In contrast, $Y_4^{\{i,i+1\}}$ may be nonplanar. Figure 4(a) shows two crossing edges; (b) shows the full graph $Y_4^{\{0,1\}}$.

As should be evident from Figure 2(c), crossing edges are rare, requiring precise placement of four points. Although it would be difficult to quantify, a "typical" $Y_4^{\{i,i+1\}}$ graph is planar.

3 Connectedness

We can see in Figure 2(a,b) that $Y_4^{\{i\}}$ is, in general, disconnected. In contrast, $Y_4^{\{i,i+1\}}$ is connected. See again Figure 2(c).

Lemma 2 $\overrightarrow{Y_4^{\{i,i+1\}}}$ *is a connected graph.*

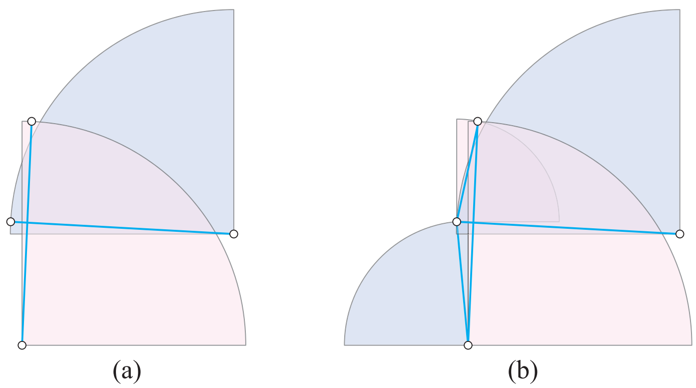


Figure 4: $Y_4^{\{0,1\}}$ can be nonplanar.

Proof: We choose i=0 w.l.o.g. So we are concerned with upward $+y$-connections, in Q_0 and Q_1. The proof is by induction on the number of points n in the set P. The basis of the induction is trivial, for an n=1 point set is connected. Let P have $n > 1$ points, and let a be the point with the lowest y-coordinate. By Assumption (2), a is unique.

Delete this from P, reducing to a point set P' with $|P'| = n-1$. Then the set of points $P' = P \setminus \{a\}$ satisfies the induction hypothesis, and so is connected into a graph $\overrightarrow{G'}$. See Figure **??**. Put back point a. Because all the quadrants determining edges $\overrightarrow{bc} \in \overrightarrow{G'}$ are Q_0 or Q_1, they lie at or above b_y, the y-coordinate of the lowest point in P', b. Thus a cannot lie in any quadrant, and so adding a to P' does not break any edge of $\overrightarrow{G'}$.[2] Finally, a itself must have at least one outgoing edge upward, for Q_0 and Q_1 cover the half-plane above a_y, which contains at least one point of P'. □

4 Undirected Spanners

It is clear that $Y_4^{\{i\}}$ is not a spanner, because it may be disconnected. Points on a negatively sloped line result in a completely disconnected graph of isolated points. Neither is $Y_4^{\{i,i+1\}}$ a spanner. Points uniformly spaced on two lines forming a 'Λ' shape both have directed paths up to the apex in $Y_4^{\{0,1\}}$, but the leftmost and rightmost lowest points can be arbitrarily far apart in the graph.

5 Directed Spanners

We turn then to directed versions of these questions. Call a directed graph a *directed spanner* if every directed path is no more than t times the path's

[2] Note that if the induction instead removed the topmost point from P, this claim would no longer hold.

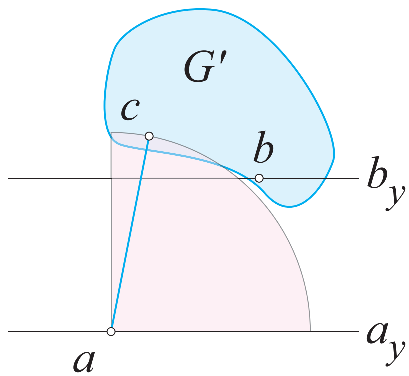


Figure 5: $Y_4^{0,1}$ must be connected.

end-to-end Euclidean distance, for t a constant.

Lemma 3 $\overrightarrow{Y_4^{\{i\}}}$ *is a directed spanner: no directed path is more than* $\sqrt{2}$ *times the end-to-end Euclidean distance.*

Proof: Let a and b be the endpoints of the path. Then the path is an xy-monotone path remaining inside $R(a, b)$. Therefore its length is at most half the perimeter of this rectangle, which is at most $\sqrt{2}$ times the diagonal length. □

Lemma 4 $\overrightarrow{Y_4^{\{i,i+1\}}}$ *is not a directed spanner: directed paths can be arbitrarily long: more than any constant* $t > 1$ *times the end-to-end Euclidean distance.*

Proof: Consider the path (a, b, c, d) in Figure 6(a). It is clear that this path can be made arbitrarily long with respect to $|ad|$, by lowering the vertical coordinates of c and d. Now we show how to avoid any other directed connection between a and d.

Let the other outgoing edge from a go to e as shown. We now direct paths from d and from e that do not connect. The idea is depicted in Figure 6(b). We create a series of nearly vertical paths from d, and from e. Above $d = (d_x, d_y)$, two points are placed at $(d_x \pm \epsilon, d_y + 1)$, $0 < \epsilon \ll 1$. The two outgoing edges from d will terminate on these. Then above those we place two more points at $(d_x \pm 2\epsilon, d_y + 2)$. Now we get both upward and diagonal connections among the four points, with one "diagonal" being horizontal.[3] The point is that all the outgoing edges are accounted for.

Repeating this construction, we can make a nearly vertical tower of points, connected by vertical paths, but otherwise insulated from one another. So the only path from a to d is (a, b, c, d). □

[3] The definition in Figure 1 shows that $(d_x - \epsilon, d_y + 1)$ will connect horizontally to $(d_x + \epsilon, d_y + 1)$.

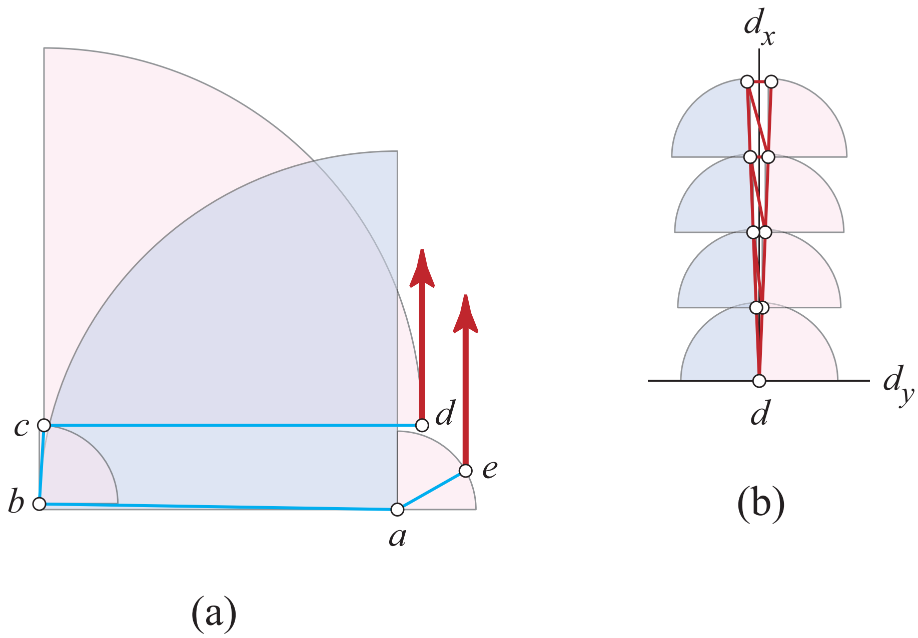


Figure 6: An arbitrarily long path in $Y_4^{\{0,1\}}$.

6 Future Work

The obvious next step is to examine properties of three quadrants, $Y_4^{\{i,i+1,i+2\}}$, before finally tackling Y_4 itself.

References

[DMP09] Mirela Damian, Nawar Molla, and Val Pincu. Spanner properties of $\pi/2$-angle Yao graphs. In *Proc. 25th European Workshop Comput. Geom.*, pages 21–24, EuroCG, March 2009.