Some Polycubes Have No Edge-Unzipping

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Abstract
It is unknown whether or not every polycube has an edge-unfolding. A polycube is an object constructed by gluing cubes face-to-face. An edge-unfolding cuts edges on the surface and unfolds it to a net, a non-overlapping polygon in the plane. Here we explore the more restricted edge-unzippings where the cut edges form a path. We construct two different polycubes neither of which has an edge-unzipping.

1 Introduction
A polycube $P$ is an object constructed by gluing cubes whole-face to whole-face, such that its surface is a manifold. Thus the neighborhood of every surface point is a disk; so there are no edge-edge nor vertex-vertex nonmanifold surface touchings. Here we only consider polycubes of genus zero. The edges of a polycube are all the cube edges on the surface, even when those edges are shared between two coplanar faces. Similarly, the vertices of a polycube are all the cube vertices on the surface, even when those vertices are flat, incident to $2\pi$ face angles. Such polycube flat vertices are degree-4. It will be useful to distinguish these flat vertices from corner vertices, non-flat vertices with incident angles $\neq 2\pi$ (degree-3, -5, or -6). For a polycube $P$, let its 1-skeleton graph $G_P$ include every vertex and edge of $P$, with vertices marked as either corner or flat.

It is an open problem to determine whether every polycube has an edge-unfolding, a tree in the 1-skeleton that spans all corner vertices (but need not include flat vertices), which, when cut, unfolds the surface to a net, a planar, non-overlapping polygon [O’R19]. Here by non-overlapping is meant that no two points, each interior to a face, are mapped to same point in the plane. This

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allows two boundary edges to coincide in the net; so the polygon is “weakly simple.” The intent is that we want to be able to cut out the net and refold to \( P \). Henceforth “edge-unfolding” will mean: an edge-unfolding to a net.

It would be remarkable if it were true that every polycube could be edge-unfolded, but no counterexample is known. There has been considerable exploration of orthogonal polyhedra, a more general type of object, for which there are examples that cannot be edge-unfolded \([BDD+98]\). (See \([DF18]\) for citations to earlier work.) But polycubes have more edges in their 1-skeleton graphs for the cut tree to follow than do orthogonal polyhedra, so it is conceivably easier to edge-unfold polycubes.

A restriction of edge-unfolding has been studied in \([DDL+10, O’R10, DDU13]\): edge-unzipping. This is an edge-unfolding whose cut tree is a path (so that the surface could be “unzipped”). It is apparently unknown if even this highly restricted edge-unzipping could unfold every polycube to a net. The result of this note is to settle this question in the negative: two different polycubes are constructed each of which has no edge-unzipping. They are shown in Figure 1 and will be described later.

Figure 1: Two polycubes that have no edge-unzipping.

2 Hamiltonian Paths

Shephard \([She75]\) introduced Hamiltonian unfoldings of convex polyhedra, what we are now calling edge-unzippings, following the terminology of \([DDL+10]\). It is easy to see that not every convex polyhedron has an edge-unzipping, simply because the rhombic dodecahedron has no Hamiltonian path. This counterexample avoids confronting the difficult non-overlapping condition. We follow a
similar strategy here, constructing a polycube with no Hamiltonian path. But there is a difference in that a polycube edge-unzipping need not include flat vertices, and so need not be a Hamiltonian path in $G_P$. Thus identifying a polycube $P$ that has no Hamiltonian path does not immediately establish that $P$ has no edge-unzipping, if $P$ has flat vertices.

So one approach is to construct a polycube $P$ that has no flat vertices—every vertex is a corner vertex. Then if $P$ has no Hamiltonian path, then it has no edge-unzipping. A natural candidate is the polycube object $P_6$ shown in Fig. 2. However, the 1-skeleton of $P_6$ does admit Hamiltonian paths, and indeed we

![Figure 2: All of $P_6$'s vertices are corner vertices.](image)

found a path that unfolds $P_6$ to a net.

Let $G_P$ be the dual graph of $P$: each cube is a node, and two nodes are connected if they are glued face-to-face. A polycube tree is a polycube whose dual graph is a tree. $P_6$ is a polycube tree. That it has a Hamiltonian path is an instance of a more general claim:

**Lemma 1** The graph $G_P$ for any polycube tree $P$ has a Hamiltonian cycle.

**Proof:** It is easy to see by induction that every polycube tree can be built by gluing cubes each of which touches just one face at the time of gluing: never is there a need to glue a cube to more than one face of the previously built object.

A single cube has a Hamiltonian cycle. Now assume that every polycube tree of $\leq n$ cubes has a Hamiltonian cycle. For a tree $P$ of $n + 1$ cubes, remove a $G_P$ leaf-node cube $C$, and apply the induction hypothesis. The exposed square face $f$ to which $C$ glues to make $P$ includes either 2 or 3 edges of the Hamiltonian cycle (4 would close the cycle; 1 or 0 would imply the cycle misses some vertices of $f$). It is then easy to extend the Hamiltonian cycle to include $C$, as shown in Figure 3.

So to prove that a polycube tree has no edge-unzipping would require an argument that confronted non-overlap. This leads to an open question:

**Question 1** Does every polycube tree have an edge-unzipping?
Figure 3: (a) $f$ contains 3 edges of the cycle (blue); (b) $f$ contains 2 edges of the cycle. The cycles are extended to $C$ by replacing the blue with the red paths.

3 Bipartite $G_P$

To guarantee the non-existence of Hamiltonian paths, we can exploit the bipartiteness of $G_P$, using Lemma 3 below.

Lemma 2 A polycube graph $G_P$ is 2-colorable, and therefore bipartite.

Proof: Label each lattice point $p$ of $\mathbb{Z}^3$ with the \{0, 1\}-parity of the sum of the Cartesian coordinates of $p$. A polycube $P$’s vertices are all lattice points of $\mathbb{Z}^3$. This provides a 2-coloring of $G_P$; 2-colorable graphs are bipartite. The parity imbalance in a 2-colored (bipartite) graph is the absolute value of the difference in the number of nodes of each color.

Lemma 3 A bipartite graph $G$ with a parity imbalance $> 1$ has no Hamiltonian path.

Proof: The nodes in a Hamiltonian path alternate colors 010101.... Because by definition a Hamiltonian path includes every node, the parity imbalance in a bipartite graph with a Hamiltonian path is either 0 (if of even length) or 1 (if of odd length).

So if we can construct a polycube $P$ that (a) has no flat vertices, and (b) has parity imbalance $> 1$, then we will have established that $P$ has no Hamiltonian path, and therefore no edge-unzipping. We now show that the polycube $P_{44}$, illustrated in Figure 4, meets these conditions.

Lemma 4 The polycube $P_{44}$’s graph $G_{P_{44}}$ has parity imbalance of 2.

Proof: Consider first the $2 \times 2 \times 2$ cube that is the core of $P_{44}$; call it $P_{222}$. The front face $F$ has an extra 0; see Fig. 5. It is clear that the 8 corners of $P_{222}$ are all colored 0. The midpoint vertices of the 12 edges of $P_{222}$ are colored 1. Finally the 6 face midpoints are colored 0. So 14 vertices are colored 0 and 12 colored 1.
Figure 4: The polycube $P_{44}$, consisting of 44 cubes, has no Hamiltonian path.

Figure 5: 2-coloring of one face of $P_{222}$. 
Next observe that attaching a cube \( C \) to exactly one face of any polycube does not change the parity: the receiving face \( f \) has colors 0101, and the opposite face of \( C \) has colors 1010.

Now, \( P_{44} \) can be constructed by attaching six copies of a 6-cube “cross,” call it \( P_+ \), which in isolation is a polycube tree and so can be built by attaching cubes each to exactly one face. And each \( P_+ \) attaches to one corner cube of \( P_{222} \). Therefore \( P_{44} \) retains \( P_{222} \)'s imbalance of 2.

The point of the \( P_+ \) attachments is to remove the flat vertices of \( P_{222} \). Note that when attached to \( P_{222} \), each \( P_+ \) has only corner vertices.

Theorem 1  There is no edge-unzipping of \( P_{44} \).

Proof: Although it takes some scrutiny of Figure 4 to verify, \( P_{44} \) has no (degree-4) flat vertices. Thus an edge-unzipping must pass through every vertex, and so be a Hamiltonian path. Lemma 4 says that \( G_{P_{44}} \) has imbalance 2, and Lemma 3 says it therefore cannot have a Hamiltonian path.

4 Construction of \( P_{14} \)

It turns out that the smaller polycube \( P_{14} \) shown in Figure 6 also has no edge-unzipping, even though it has flat vertices. To establish this, we still need an imbalance > 1, which easily follows just as in Lemma 4.

Figure 6: \( P_{14} \): \( P_{222} \) with six 1-cube attachments.
Lemma 5  The polycube $P_{14}$’s graph $G_{P_{14}}$ has parity imbalance of 2.

But notice that $P_{14}$ has three flat vertices: $a$, $b$, and $c$.

Theorem 2  There is no edge-unzipping of $P_{14}$.

Proof: An edge-unzipping need not pass through the three flat vertices, $a$, $b$, and $c$, but it could pass through one, two, or all three. We show that in all cases, an appropriately modified subgraph of $G_{P_{14}}$ has no Hamiltonian path. Let $\rho$ be a hypothetical edge-unzipping cut path. We consider four exhaustive possibilities, and show that each leads to a contradiction.

(0) $\rho$ includes $a, b, c$. So $\rho$ is a Hamiltonian path in $G_{P_{14}}$. But Lemma 5 says that $G_{P_{14}}$ has imbalance 2, and Lemma 3 says that no such graph has a Hamiltonian path.

(1) $\rho$ excludes one flat vertex $a$ and includes $b, c$. (Because of the symmetry of $P_{14}$, it is no loss of generality to assume that it is $a$ that is excluded.) If $\rho$ excludes $a$, then it does not travel over any of the four edges incident to $a$. Thus we can delete $a$ from $G_{P_{14}}$; say that $G_{-a} = G_{P_{14}} \setminus a$. This graph is shown in Fig. 7. Following the coloring in Fig. 5, all corners of $P_{222}$ are colored 0, so each of the edge midpoints $a, b, c$ is colored 1. The parity imbalance of $P_{14}$ is 2 extra 0’s. Deleting $a$ maintains bipartiteness and increases the parity imbalance of $G_{-a}$ to 3. Therefore by Lemma 5 $G_{-a}$ has no Hamiltonian path, and such a $\rho$ cannot exist.

(2) $\rho$ includes just one flat vertex $c$, and excludes $a, b$. (Again symmetry ensures there is no loss of generality in assuming the one included flat vertex is $c$.) $\rho$ must include corner $x$, which is only accessible in $G_{P_{14}}$ through the three flat vertices. If $\rho$ excludes $a, b$, then it must include the edge $cx$. Let $G_{-ab} = G_{P_{14}} \setminus \{a, b\}$. In $G_{-ab}$, $x$ has degree 1, so $\rho$ terminates there. It must be that $\rho$ is a Hamiltonian path in $G_{-ab}$, but the deletion of $a, b$ increases the parity imbalance to 4, and so again such a Hamiltonian path cannot exist.

(3) $\rho$ excludes $a, b, c$. Because corner $x$ is only accessible through one of these flat vertices, $\rho$ never reaches $x$ and so cannot be an edge-unzipping.

Thus the assumption that there is an edge-unzipping path $\rho$ for $P_{14}$ reaches a contradiction in all four cases. Therefore, there is no edge-unzipping path for $P_{14}$.

5  Edge-unfoldings of $P_{14}$ and $P_{44}$

Now that it is known that $P_{14}$ and $P_{44}$ each have no edge-unzipping, it is natural to wonder if either settles the edge-unfolding open problem: Can they...
Figure 7: Schlegel diagram of $G_{-a}$. We follow [DF18] in labeling the faces of a cube as $F, K, R, L, T, B$ for Front, Back, Right, Left, Top, Bottom respectively. The corners of $P_{222}$ are labeled $0, 1, 2, 3$ around the bottom face $B$, and $4, 5, 6, 7$ around the top face $T$. $m$ is the vertex in the middle of $B$. The edges deleted by removing $a$ are shown dashed.
be edge-unfolded? Indeed both can: see Figures 8 and 9. The colors in these layouts are those used by Origami Simulator [GDG18]. Figure 10 shows a partial folding of $P_{44}$, and animations are at [http://cs.smith.edu/~jorourke/Unf/NoEdgeUnzip.html](http://cs.smith.edu/~jorourke/Unf/NoEdgeUnzip.html).

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**References**


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3 Just to verify this conclusion, we constructed these graphs in Mathematica and `FindHamiltonianPath[]` returned {} for each.
Figure 9: Edge-unfolding of $P_{44}$. Colors: green=cut, red=mountain, blue=valley, yellow=flat.
Figure 10: Partial folding of the layout in Fig. 9. Compare Fig. 4.


