Unfolding Polyhedra

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Abstract

Starting with the unsolved “Dürer’s problem” of edge-unfolding a convex polyhedron to a net, we specialize and generalize (a) the types of cuts permitted, and (b) the polyhedra shapes, to highlight both advances established and which problems remain open.

1 Introduction

Dürer’s problem asks whether every convex polyhedron may be cut along edges and unfolded to a single non-overlapping simple polygon in the plane, a net [DO07] [O’R13]. This is attributed to Dürer because he drew many such unfoldings ca. 1500, although the question was not formulated mathematically until 1975 [She75]. It remains open, although there has been recent (minor) progress [O’R18] [O’R17]. Here we survey several generalizations and specializations of this central problem, emphasizing what is settled and what remains unresolved.

Unfolding the surface of a polyhedron to a single, flat piece in the plane requires that the cuts form a spanning tree of the vertices. We classify cuts in four types $C$:

1. edge-unfold: All cuts are polyhedron edges, as in Dürer’s problem.

2. anycut-unfold: The cuts may be generalized to any curve on the surface that form a spanning tree of the vertices.$^1$

3. edge-unzip: The cut edges form a Hamiltonian path of the 1-skeleton. This natural specialization was introduced by Shephard [She75] and explored as “unzipping” in [DDL+10].$^2$ Most classes of polyhedra do not admit edge-unzippings [DDEO19].

4. anycut-unzip: The cuts form a simple curve on the surface that includes every vertex. So a generalization (anycut) of a specialization (unzipping).

The second classification we explore varies the shapes $P$ of the polyhedra:

1. convex polyhedra: All faces convex, all dihedral angles $\leq \pi$, as in Dürer’s problem.

2. spherical polyhedra: Specializing that all vertices lie on a sphere [O’R15].$^3$

3. nonconvex polyhedra. A broad generalization, and where most applications lie.

4. orthogonal polyhedra form an important subclass of nonconvex polyhedra [BDD+98] [O’R08] [DFO07] [DDFO17]. All faces meet at right angles.

5. polycubes: Polyhedra built by gluing unit cubes whole-face to whole-face. Here all cube edges, even those with dihedral angle $\pi$, are available for cutting. So these are potentially easier to edge-unfold than are orthogonal polyhedra [RALSZ19].

For each class of polyhedra $P$, and each type of cuts $C$, we can ask:

Can every polyhedron in $P$ be $C$-unfolded to a net?

The status of these $4 \times 5 = 20$ questions is summarized in Table 1: 6 are unresolved.

References


$^1$“Anycut” is new terminology, intended to replace the “general unfoldings” in [DO07].

$^2$“Unzipping” is my slight variation on their “zipper unfoldings.”

$^3$“Spherical” is often called inscribed in the literature, and “spherical polyhedra” are sometimes tilings of the sphere by geodesic arcs.
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<th>Edge-Unf</th>
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<td>Polycubes</td>
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Table 1: Open Problems: ?=open, ✓=proven true, ×=counterexamples.


