Computational Geometry Column 48

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Abstract

A new class of art gallery -like problems inspired by wireless localization is discussed.

An interesting new variant of classic art gallery theorems has emerged from wireless localization, as described in a paper by Eppstein, Goodrich, and Sitchinava [EGS06]. The geometric problem may be phrased as follows. A "virtual" polygon P of n vertices is given. A number g of point stations are placed in the plane, each of which broadcasts a unique index (or "key") within a fixed angular range. P is virtual in the sense that it does not block broadcasts. The goal is to place and orient stations so that each point in the plane can determine if it is in or out of P from a monotone Boolean formula (AND $(\cdot)$ OR $(+)$ operations only) composed from the broadcasts. For example, the pentagon in Fig. 1a is determined by the formula $A \cdot B \cdot C$, and the 12-vertex polygon in Fig. 1b is determined by $A \cdot B \cdot F + D \cdot C \cdot E + D \cdot C \cdot F$.

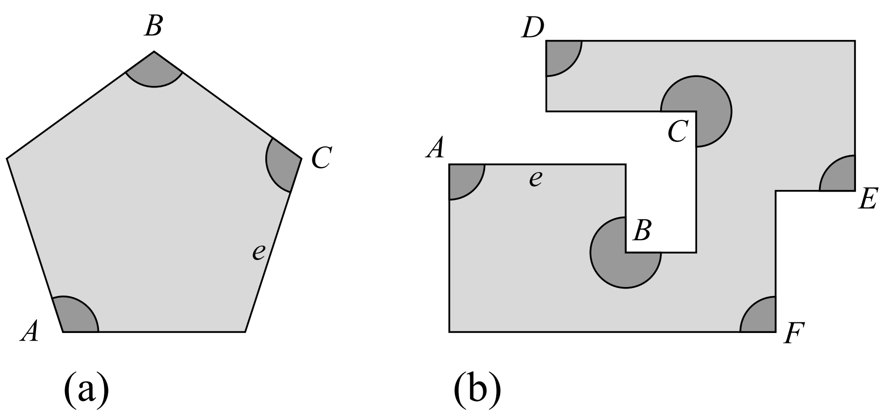

Figure 1: (a) Convex 5-gon; (b) Orthogonal polygon, $n = 12$.

The stations might be angular RF, IR, or laser transmitters. A sensor at a point in the plane receives the keys from those stations that angularly cover it, and can determine if it is in P from the Boolean formula. The primary goal is to minimize g, the number of stations, or "guards" in analogy with art gallery theorems.

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Aside from the connection to art gallery theorems, work on floodlights (e.g., [ST03]) and CSG Peterson formula (e.g., [DGHS93]) is relevant, but none of the related work solves the problem as posed. For example, one 180° station per polygon edge can achieve $g = n$ [DGHS93], but $g < n$ is desirable.

Tight bounds have been achieved for two special classes of polygons: $g = \lceil n/2 \rceil$ stations suffice, and are sometimes necessary, for convex polygons and for orthogonal polygons. In both cases, a station at every other vertex, broadcasting over the internal angle there, achieves the goal. For convex polygons, it is clear this suffices. Necessity of $\lceil n/2 \rceil$ for any polygon without collinear edges is established by observing that fewer stations implies that some edge e of P is not collinear with a broadcast boundary line, leaving a neighborhood of e with ambiguous in/out status. Removing station C in Fig. 1a places e in this status.

Let e be a horizontal top edge of an orthogonal polygon. Their algorithm for orthogonal polygons places a station at e's left endpoint, and then a station at every other vertex after that; see Fig. 1b. They prove this suffices by a clever induction.

So far we have employed what they call "natural angle guards": stations at a vertex of P covering the vertex's interior angle. For general polygons, such stations do not suffice. Even

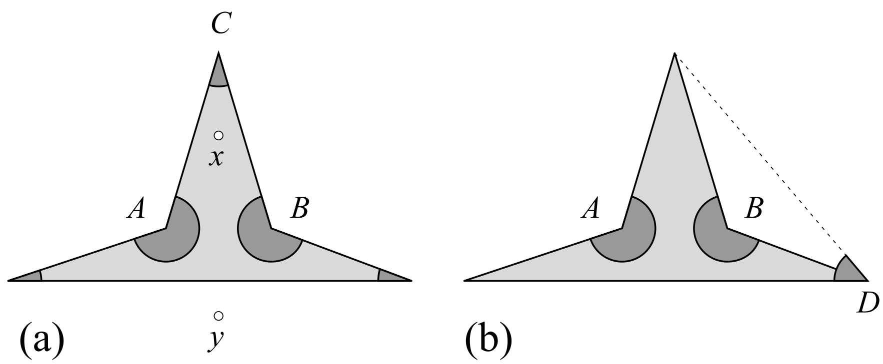

Figure 2: (a) Internal angle stations do not suffice; (b) Coverage by three stations: formula $A{\cdot}B{\cdot}D$.

one at every vertex of the pentagon shown in Fig. 2a fails to distinguish between points x and y, who both receive $A{\cdot}B{\cdot}C$. Fig. 2b shows that three stations do suffice for this pentagon, but one, D, broadcasts both inside and outside P. Indeed they establish through a nontrivial argument involving partitioning P into quadrilaterals, pentagons, and at most one hexagon, that $n-2$ stations at vertices always suffice, providing the best upper bound on the general problem. When edge collinearities are permitted, there is a lower bound of $g = \Omega(\sqrt{n})$ and orthogonal polygon examples where $O(\sqrt{n})$ stations suffice. But the best lower bound for general position polygons is the $\lceil n/2 \rceil$ necessity mentioned previously. The considerable gap between the $\lceil n/2 \rceil$ and $n-2$ bounds remains to be closed.

Finally, it is desirable not only to minimize the number of stations, but also to achieve *concise* formulas, those, when in DNF, have a constant number of terms in each conjunction. For then a sensor can prove it is in P with an $O(1)$-size "certificate." The formula for convex polygons is not concise, as it is a conjunction of $\lceil n/2 \rceil$ terms. Tradeoffs have been established between the competing goals in some cases. For example, conjunctive terms for a convex polygon can be kept to $O(1/\epsilon)$ length, for any constant $\epsilon > 0$, at the cost of inflating g to $\lceil n/2 \rceil (1 + \epsilon)$.

References

[DGHS93] D. P. Dobkin, L. Guibas, J. Hershberger, and J. Snoeyink. An efficient algorithm for finding the CSG representation of a simple polygon. *Algorithmica*, 10:1–23, 1993.

[EGS06] D. Eppstein, M. T. Goodrich, and N. Sitchinava. Guard placement for wireless localization. arXiv: cs.CG/0603057, 2006.

[ST03] B. Speckmann and C. D. Tóth. Allocating vertex π-guards in simple polygons via pseudo-triangulations. In *Proc. 14th Annu. ACM-SIAM Sympos. Discrete Algorithms*, pages 109–118, Philadelphia, 2003.