Computational Geometry Column 46

Joseph O'Rourke*

Abstract

The old problem of determining the chromatic number of the plane is revisited.

The question of the chromatic number of the Euclidean plane $\mathbb{E}^2$ has been unresolved for over fifty years. Informally, the question asks: How many colors are needed to paint the plane so that no two points a unit distance apart are painted the same color? If the same question is asked of the line, the answer is 2: Coloring $[0, 1)$ red, $[1, 2)$ blue, etc., ensures that no two unit-separated points have the same color. Here I report on a few new developments, and some related open problems that are perhaps easier.

One can view the question as asking for the chromatic number $\chi(\mathbb{E}^2)$ of the infinite *unit-distance graph* G, with every point in the plane a node, and an arc between two nodes if they are separated by a unit distance. Erdős and de Bruijn showed [EdB51] that the chromatic number of the plane is attained for some finite subgraph of G. This result led to narrowing the answer to $4 \le \chi(\mathbb{E}^2) \le 7$. For example, the lower bound of 4 is established by the "Moser graph" shown in Fig. 1, which needs 4 colors.

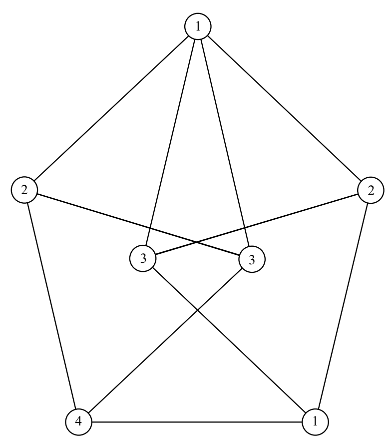

Figure 1: All edge lengths are 1. Four colors are needed to color the nodes so that no two adjacent nodes have the same color.

However, the Erdős-de Bruijn compactness argument depends crucially on the Axiom of Choice. Recently Shelah and Soifer [SS03] showed that the chromatic number of the plane

*Dept. of Computer Science, Smith College, Northampton, MA 01063, USA. orourke@cs.smith.edu.
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may depend on which axioms of set theory one employs. In particular, they prove that if every finite unit-distance graph can be 4-colored, then the chromatic number of the plane is 4 under the standard Zermelo-Fraenkel axioms with the Axiom of Choice (ZFC), as one would expect. But if instead one uses ZF and a weaker axiom of "dependent choices," and further assumes that every set of real numbers is Lebesgue measurable (roughly: has an area), then the chromatic number of the plane must be strictly greater than 4.

This problem is difficult enough to have a $1000 reward promised for its solution by Ron Graham, who is continuing the Erdős tradition of tagging open problems with monetary awards proportional to their perceived difficulty [Gra03]. Here I report on two related problems of Graham, which may be classified as "Euclidean Ramsey problems" [Gra04a] [Gra04b].

Let T be a triangle in the plane, with each point of the plane assigned a color. T is *monochromatic* if its three vertices are painted the same color. Now we imagine congruent copies of T moved around the plane via rigid motions, and seek a spot where T is monochromatic.

Conjecture 1 ($50) *For any triangle T, there is a 3-coloring of the plane with no monochromatic copy of T.*

Note here the coloring may depend on T.

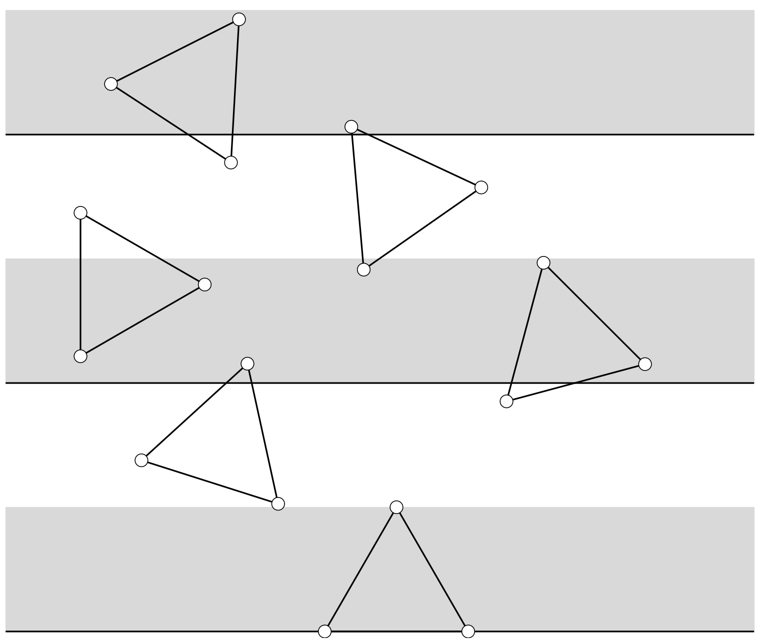

Figure 2: Half-open strips of width $\sqrt{3}/2$ preclude a monochromatic copy of the illustrated unit equilateral triangle.

It is especially interesting to consider the unit edge-length equilateral triangle T_1, which is a subgraph of the unit-distance graph G. Conjecture 1 suggests that a 3-coloring can avoid a monochromatic copy, but in fact a 2-coloring suffices. Paint the plane in half-open alternating strips of width $\sqrt{3}/2$. As Fig. 2 shows, T_1 has no monochromatic position, just barely failing when two vertices are placed on the lower closed boundary of a strip. The surprising conjecture is that the equilateral triangle is very special, in that, for *any* non-equilateral triangle T, *every* 2-coloring admits a monochromatic copy of T:

Conjecture 2 ($100) *Every 2-coloring of the plane contains a monochromatic copy of every triangle, except possibly for a single equilateral triangle.*

This is known to be true for several classes of triangles, for example right triangles [Sha76]. So, for example, the same strip coloring captures every right triangle; see Fig. 3.

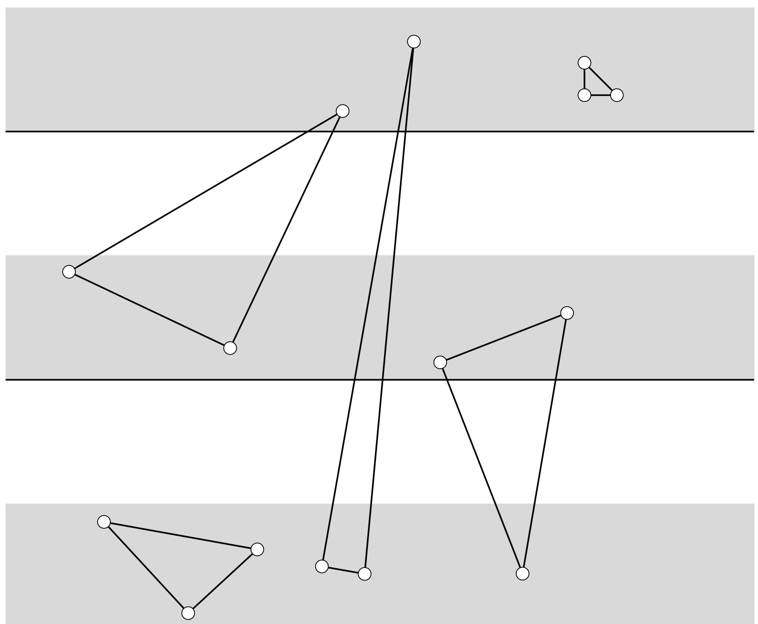

Figure 3: Every right triangle can be positioned to have its vertices in one color class.

All these notions generalize to arbitrary dimensions. In 3D, it is known that every 3-coloring of $\mathbb{E}^3$ includes a monochromatic copy of any right triangle. The knowledge gap for the chromatic number of space is even wider than for the plane: it is only known to satisfy $6 \leq \chi(\mathbb{E}^3) \leq 15$. See [Gra04a] [Gra04b] for further results and references.

References

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