

# Unfolding Polyhedra

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## Abstract

Starting with the unsolved “Dürer’s problem” of edge-unfolding a convex polyhedron to a net, we specialize and generalize (a) the types of cuts permitted, and (b) the polyhedra shapes, to highlight both advances established and which problems remain open.

## 1 Introduction

*Dürer’s problem* asks whether every convex polyhedron may be cut along edges and unfolded to a single non-overlapping simple polygon in the plane, a *net* [DO07] [O’R13]. This is attributed to Dürer because he drew many such unfoldings ca. 1500, although the question was not formulated mathematically until 1975 [She75]. It remains open, although there has been recent (minor) progress [O’R18] [O’R17]. Here we survey several generalizations and specializations of this central problem, emphasizing what is settled and what remains unresolved.

Unfolding the surface of a polyhedron to a single, flat piece in the plane requires that the cuts form a spanning tree of the vertices. We classify cuts in four types  $\mathcal{C}$ :

1. *edge-unfold*: All cuts are polyhedron edges, as in Dürer’s problem.
2. *anycut-unfold*: The cuts may be generalized to any curve on the surface that form a spanning tree of the vertices.<sup>1</sup>
3. *edge-unzip*: The cut edges form a Hamiltonian path of the 1-skeleton. This natural specialization was introduced by Shephard [She75] and explored as “unzipping” in [DDL<sup>+</sup>10].<sup>2</sup> Most classes of polyhedra do not admit edge-unzippings [DDEO19].
4. *anycut-unzip*: The cuts form a simple curve on the surface that includes every vertex. So a generalization (anycut) of a specialization (unzipping).

The second classification we explore varies the shapes  $\mathcal{P}$  of the polyhedra:

1. *convex polyhedra*: All faces convex, all dihedral angles  $\leq \pi$ , as in Dürer’s problem.

2. *spherical polyhedra*: Specializing that all vertices lie on a sphere [O’R15].<sup>3</sup>
3. *nonconvex polyhedra*. A broad generalization, and where most applications lie.
4. *orthogonal polyhedra* form an important subclass of nonconvex polyhedra [BDD<sup>+</sup>98] [O’R08] [DFO07] [DDFO17]. All faces meet at right angles.
5. *polycubes*: Polyhedra built by gluing unit cubes whole-face to whole-face. Here all cube edges, even those with dihedral angle  $\pi$ , are available for cutting. So these are potentially easier to edge-unfold than are orthogonal polyhedra [RALSZ19].

For each class of polyhedra  $\mathcal{P}$ , and each type of cuts  $\mathcal{C}$ , we can ask:

Can every polyhedron in  $\mathcal{P}$  be  $\mathcal{C}$ -unfolded to a net?

The status of these  $4 \times 5 = 20$  questions is summarized in Table 1: 6 are unresolved.

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<sup>1</sup> “Anycut” is new terminology, intended to replace the “general unfoldings” in [DO07].

<sup>2</sup> “Unzipping” is my slight variation on their “zipper unfoldings.”

<sup>3</sup> “Spherical” is often called *inscribed* in the literature, and “spherical polyhedra” are sometimes tilings of the sphere by geodesic arcs.

<i>Shapes</i>	<i>Edge-Unf</i>	<i>(Specialize)</i> <i>Edge-Unzip</i>	<i>(Generalize)</i> <i>Anycut-Unf</i>	<i>(Gen/Spec)</i> <i>Anycut-Unzip</i>
Convex Polyh	?	✗	✓	?
Spherical	?	✗	✓	?
Nonconvex Polyh	✗	✗	?	✗
Orthogonal	✗	✗	✓	✓
Polycubes	?	✗	✓	✓

Table 1: Open Problems: ?=open, ✓=proven true, ✗=counterexamples.

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