Thickness-2 Box Complexes are 3-Colorable

The Coloring Clique:
Lily, Jessica, Micaela, Emily, Joe, Victoria, Rawia, Stephanie*

February 20, 2011

Abstract

Definition and proofs. Pretty much finished now.

1 Definitions

This proof only works on a restricted class of objects built from boxes. First we will define the class of objects.

The Objects. Each element of the object is a rectangular box. The boxes are glued together whole-face-to-whole-face to form a *box complex*. We do not allow two boxes sharing only part of a face. However, it is possible for two boxes to share just part of an edge, or share only one vertex. See Figure 1.

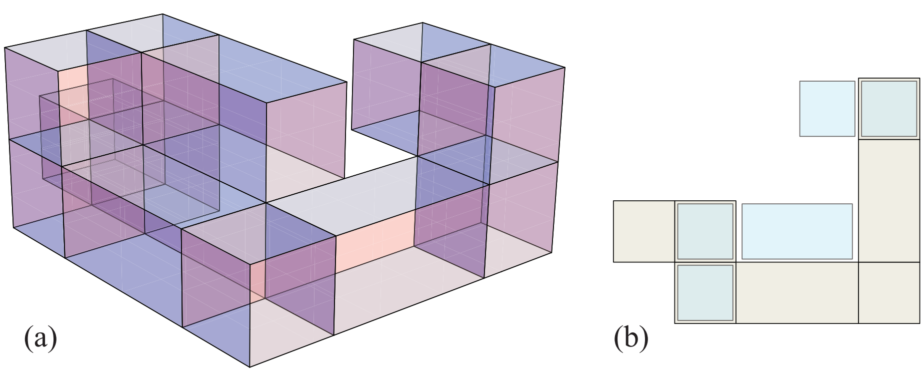

Figure 1: (a) A box complex. (b) Overhead map of complex: layer 1 (tan), layer 2 (blue). Two boxes in layer 2 are "suspended."

*Departments of Mathematics and of Computer Science, Smith College, Northampton, MA 01063, USA.

In two dimensions, the analogous object is a *rectangle complex*; Figure 2(a). In this proof, we will need a relaxation of the whole-edge-to-whole-edge gluing in a rectangle complex, to permit two rectangles to share just part of an edge. Let us call this looser conglomeration a *rectangle collection*; Figure 2(b).

In all these objects, the rectangles or box edges are parallel to orthogonal xyz Cartesian axes, and all vertices have integer coordinates, i.e., all corners lie on the integer lattice, $\mathbb{Z}^2$ or $\mathbb{Z}^3$ respectively.

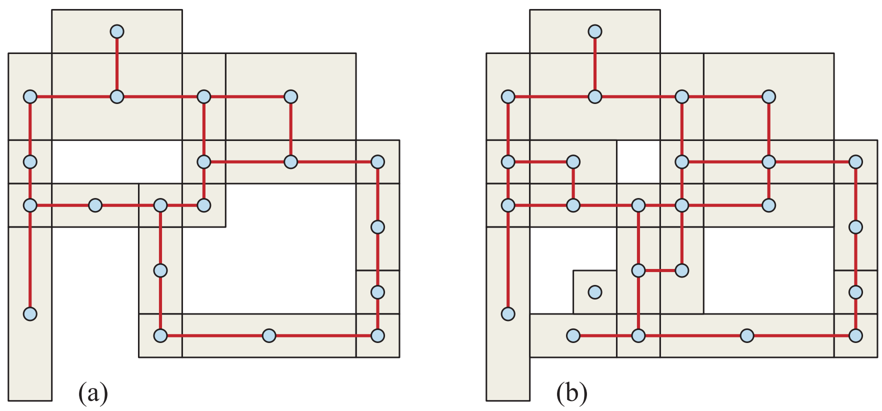

Figure 2: (a) A rectangle complex and its dual graph. (b) A rectangle collection and its dual graph.

Dual Graph. Each object has an associated dual graph that forms the basis of coloring rules. For a rectangle complex, the nodes of the dual graph are rectangles, and there is an arc between each pair of rectangles that share a (whole) edge (and they can only share a whole edge in a rectangle complex). For a rectangle collection, the nodes of the dual graph are rectangles, and again there is an arc between each pair of rectangles that share a whole edge. Note that none of the partial-edge contacts, which are permitted in a rectangle collection, result in an arc in the dual graph. Finally, the dual graph of a box complex has a node for each box, and an arc between each pair of boxes that share a (whole) face (and they can only share a whole face in a box complex). Two boxes that share a whole edge, or part of an edge, or just one vertex, but not a face, are not connected by an arc in the dual graph. See again Figure 2.

Coloring. An object is *solid-colored* by coloring the nodes of the dual graph so that no two adjacent nodes are assigned the same color. Thus abutting rectangles or boxes must get different colors. We call it "solid"-coloring because the entire solid box is given a color, which must be different from any other box that shares a face with it. A class of objects is said to be *k-colorable* if k colors always suffice for any object in that class. Our goal is to prove sharp colorability results of the form: a certain class of objects is k-colorable, and sometimes k colors are needed.

2 2D Theorems

Theorem 1 *A rectangle complex is 3-colorable, and sometimes 3 colors are needed.*

Proof: The proof is by induction. For the base case, a one-rectangle complex can be colored with one color. Let $\mathcal{C}$ be a complex of n rectangles. We are going to identify a *corner rectangle*: a rectangle with two adjacent sides *exposed*. A corner rectangle has degree 2 in the dual graph G.

Place a minimal bounding box around the complex. One or more rectangles touch the top of this bounding box; otherwise the top of the box could be lowered, and it would not be minimal. Let r be the rightmost of all the rectangles touching the top edge of the bounding box. Because r is rightmost, it cannot be adjacent to a rectangle glued to its right edge. And its top edge must also be exposed. Thus it has degree at most 2 in G.

Remove r from $\mathcal{C}$, and let $\mathcal{C}' = \mathcal{C} \setminus r$ be the resulting complex of $n-1$ rectangles. $\mathcal{C}'$ is 3-colorable by the induction hypothesis. Now put r back, reforming $\mathcal{C}$, and color it with a color not used by its at most two adjacent rectangles. We have colored $\mathcal{C}$ with three colors. $\square$

Theorem 2 *A rectangle collection is 3-colorable, and sometimes 3 colors are needed.*

Proof: The proof here is nearly the same as the previous proof. Again we identify r as the rightmost of the topmost rectangles. Now we cannot conclude that r is exposed to the right in the sense that there is no rectangle touching it to its right: it could be that there is a rectangle r' sharing just part of r's right side. But then r and r' are not adjacent in the dual graph G, because only whole-side sharing results in adjacency in G. So still r has at most degree 2 in G, which is the only property that the previous relied upon. So the argument goes through as before: remove r, 3-color by induction, put r back and color it with a color that does not clash with its at most two neighbors. $\square$

3 Thickness-2 Theorem

Say that a box complex has *thickness-t* if its height in one of the three directions (which we take to be the z-direction) is $\leq t$, i.e., it lies between two parallel planes separated by distance at most t, and it does not lie between closer parallel planes. A box complex of thickness-1 has the same dual graph as a rectangle complex: because we restrict all coordinates to integers, boxes must be at least one-unit in thickness, so the rectangle bases of the boxes has the same dual graph as the boxes themselves. Thus a box complex of thickness-1 is 3-colorable by Theorem 1. The first interesting case is thickness-2:

Theorem 3 *A box complex of thickness-2 is 3-colorable, and sometimes 3 colors are needed.*

Proof: That 3 colors are sometimes needed is immediate, because a box complex of thickness-1 sometimes needs 3 colors, and such a complex could be stretched by a factor of 2 in the z-direction.

Let $\mathcal{C}$ be the rectangle complex, and G its dual graph. Call a box *tall* if it has z-height 2. First notice that the set of tall boxes forms a separate component of G, call it G_2. This is because no box of height 1 can be adjacent to a tall box. Just as we argued above (for thickness-1 complexes), G_2 is the same as the dual graph of the rectangles that form the bases of all the tall boxes. Thus G_2 is 3-colorable by Theorem 1.

Henceforth we concentrate solely on the box complex $\mathcal{C}_1$ consisting of all the boxes of height 1, and its dual graph G_1. Although the boxes have z-height 1, they can have any (integer) size in the x and y dimensions. Without loss of generality, assume all the boxes in $\mathcal{C}_1$ lie between $z = 0$ and $z = 2$. Call the collection of boxes that touch $z = 0$ to be in *layer 1* and those that touch $z = 2$ to be in *layer 2*. (See again Figure 1.) Finally, say that a box in layer 2 that has no box beneath it in layer 1, is *suspended*.

The proof now takes four steps:

1. Under every suspended box in layer 2, add an identical *imaginary* box in layer 1.
2. Argue that the boxes of $\mathcal{C}_1$ in layer 1, plus the imaginary boxes, form a box collection $\mathcal{C}_1'$.
3. Apply Theorem 2 to conclude that $\mathcal{C}_1'$ is 3-colorable.
4. Color every box in layer 2 with color $c + 1 \pmod 3$ if the box of $\mathcal{C}_1'$ underneath it is colored c.

Now we detail these four steps.

1. Add imaginary boxes in layer 1 under every suspended box in layer 2.

   Note that by our definition of a box complex, it is not possible for, say, a large box in layer 2 to have a small box underneath it, because then we would have two boxes touching in a part of rather than in a whole face. So every box b_2 in layer 2 either has an identical box b_1 beneath it in layer 1, or it has nothing beneath it at all. In the former case, we do nothing. In the latter case, we add a "imaginary" box b_1' underneath it. b_2 and b_1' share a face (at $z = 1$), and so are adjacent in G_1. See Figure 3.

2. Let $\mathcal{C}_1'$ be the set of boxes from $\mathcal{C}_1$ in layer 1, plus all the imaginary boxes added in Step 1. $\mathcal{C}_1'$ is a collection of boxes, as defined previously. (In fact, the definition of a collection of boxes was designed precisely to capture $\mathcal{C}_1'$.)

3. $\mathcal{C}_1'$ is 3-colorable. This follows immediately from Theorem 2.

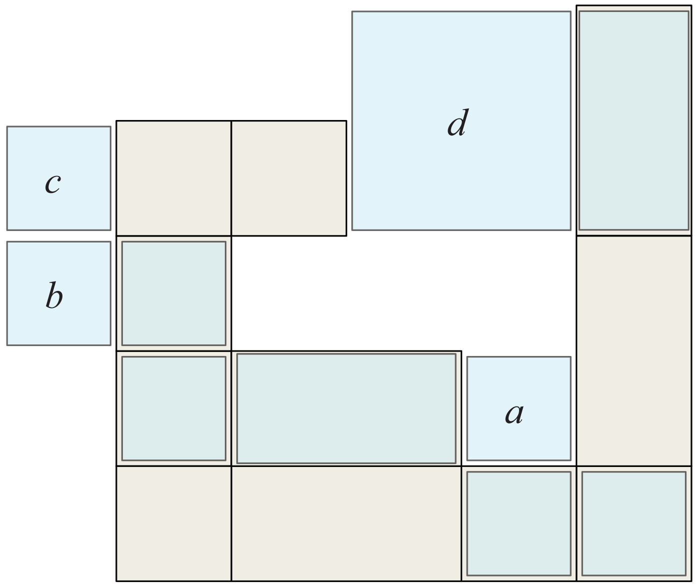

Figure 3: Boxes a, b, c, d on level 2 (blue) are suspended. Note that the imaginary boxes underneath a and d meet boxes on level 1 (tan) in subparts of faces, and so form a box collection on level 1 rather than a box complex.

4. Let the colors used to solid-color $\mathcal{C}_1'$ be $c \in \{0, 1, 2\}$. Let b_2 be a box in layer 2. Note that it must have a box of $\mathcal{C}_1'$ beneath it, either a box of $\mathcal{C}_1$ or an imaginary box. Call that box b_1. It was colored in Step 3. If b_1 is assigned color c, then color b_2 with color $c + 1 (\bmod 3)$.

   All of layer 1 is colored properly by the $\mathcal{C}_1$ subset of $\mathcal{C}_1'$. This color assignment avoids color-conflicts between layers by the $c + 1$ rule. Because every box in layer 2 gets its color from a box of $\mathcal{C}_1'$ in layer 1, and $\mathcal{C}_1'$ is properly colored, layer 2 is properly colored with just a shift in the colors by $+1$. In more detail, if two boxes a_2 and b_2 in layer 2 are adjacent, then there are two boxes a_1 and b_1 beneath them in $\mathcal{C}_1'$ which receive different colors; and so a_2 and b_2 receive different colors.

$\square$

In some sense this proof says that a box complex of thickness-2 has the same structure as a planar rectangle complex, and so is 3-colorable.