Russell’s Paradox and Frege’s Mistake

Why are logicians interested in paradoxes? Partly because paradoxes challenge our reason. A paradox unresolved is an affront to our understanding of the concepts involved. Paradoxes almost always teach us something about those concepts. The existence of a paradox shows that we have not grasped all there is to grasp about certain concepts and the logic that deals with them. Furthermore, paradoxes are, like puzzles, fun. Finally, the discipline of modern logic arose from a paradox called Russell’s paradox. The story behind Russell’s paradox is the guiding myth or folktale of modern logic, and we want to share it with you.

Modern logic was discovered by Gottlob Frege, a German mathematician and philosopher (1848-1925). It was he who discovered how to manipulate quantifiers (\(\forall\) and \(\exists\)), truth functions (\(\neg, \land, \text{ and } \Rightarrow\)), variables, and predicates. It was Frege who taught us how to formalize using the logical symbols, and it was he who taught us the essential rules for drawing conclusions, such as those used to derive the contradiction implicit in the barber’s paradox. It is not unreasonable to say that Frege was the greatest logician who ever lived. He took a subject that had been essentially unchanged since its codification by Aristotle over 2000 years ago and completely revolutionized it, just as Einstein revolutionized physics and Darwin revolutionized biology.

Unfortunately, Frege’s genius was largely unrecognized in his lifetime. His standards were so high, his ideas so novel, his symbolization so extraordinary for the time that few understood him. It did not help his cause that he scheduled his lectures for 7:00 a.m. and mumbled them while facing the blackboard (the lectures were not well attended). Nevertheless, among his few students and correspondents were the titans of early twentieth-century logic, philosophy, and mathematics.

Still, Frege had a plan. He was not interested in logic simply for its own sake but had in mind resolving some ancient philosophical problems. What is mathematics? How is knowledge of mathematics possible? The standard answers to these questions did not satisfy Frege. Some people say that mathematics deals with ideas, but Frege wondered how it is that my idea of 4 agrees with your idea of 4. Some people say that mathematics is a game played with formal symbols, but Frege wondered why a game should be so useful and why we should feel so bound by mathematical results if mathematics is indeed a game. Finally, some people believe that there is a mysterious realm of mathematical objects for which human beings have a special
mental faculty, a sort of ESP, for intuiting those objects. Frege thought this belief was a superstition.

Frege had his own answer to these questions; in essence, mathematics is nothing but logic, he said. If one takes the symbols and rules of modern logic (the logic Frege created) and adds just one subtle touch, the concept of set, then all of mathematics follows. That is, according to Frege, every concept of mathematics can be explicitly defined in terms of logic, and every statement of mathematics can be translated by a wff of logic. The rules of mathematics are just the logical rules described in “Deduction.” Finally, all of the basic principles (axioms) of mathematics can be derived from the fundamental laws of logic.

These were amazing assertions. The Western philosophical tradition had rarely seen such surprising claims expressed so precisely. Yet Frege set about demonstrating these claims. He did not argue about them the way philosophers usually do (with plenty of logical leaps) but attempted to prove them line by line, the way mathematicians do! Frege might not have been popular, but he was surely supported by the rightness of his cause. When he deduced mathematics from logic, he figured, then his procedures would be vindicated. After writing an initial book outlining his project, The Foundations of Arithmetic, he began work on a two-volume set in which the project was developed in detail, The Fundamental Laws of Arithmetic.

A key concept of Frege’s is that of a set. We discussed sets in Chapter 4. To that discussion we add some notation. To signify that \( b \) is an element of set \( c \) we will write:

\[ b \in c. \]

The symbol “\( \in \)” is really a 2-place predicate but it’s customary to write \( b \in c \) instead of \( \in bc \).

Frege added only two principles to basic logic. One simply said that sets with the same members were the same sets (e.g., \( \{ \text{Jim, Tom} \} = \{ \text{Tom, Jim} \} \)). More generally,

\[ x = y \iff \forall z(z \in x \iff z \in y). \]

The other principle asserted that every property determined a set: Given any property (1-place predicate) \( F \), there was a set consisting of all those things that had \( F \).
In symbols, for each $F$, 

$$\exists y \forall x (x \in y \Leftrightarrow Fx).$$

With those two properties in place, Frege began his detailed derivation of mathematics from the basic principles of logic.

Just as the second volume of *Fundamental Laws of Arithmetic* was going to press, however, Frege received a letter from the young British philosopher and logician Bertrand Russell. Russell’s letter called Frege’s attention to the axiom of set existence (For each $F$, $\exists y \forall x (x \in y \Leftrightarrow Fx)$) and asked what happens when we consider $F$ to be the property “is not a member of itself,” that is, $Fx \Leftrightarrow \neg(x \in x)$. Some sets seem to belong to themselves, like the set of all sets and the set of all abstract objects. But many sets don’t belong to themselves; for example, the set of all women is not a woman, and the set $\{\text{Tom, Jim}\}$ is neither Tom nor Jim. According to Frege’s principle, there should be a set consisting of all and only those sets that don’t belong to themselves:

$$\exists y \forall x (x \in y \Leftrightarrow \neg(x \in x)).$$

But if it’s true for all $x$ that

$$x \in y \Leftrightarrow \neg(x \in x),$$

then it’s true in particular for $y$:

$$y \in y \Leftrightarrow \neg(y \in y).$$

This is impossible.

Thus, Russell showed that Frege’s “basic principle of logic” was not a principle of logic. In fact, it was not even true. When Frege realized the gap in his foundation for arithmetic, he said, “Arithmetic totters.” His life’s work lay in ruins. Years later, Russell wrote about Frege’s response to this situation.
As I think about acts of integrity and grace, I realise there is nothing in my knowledge to compare with Frege’s dedication to truth. His entire life’s work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.

Such is Frege’s tragic story, and it reveals both the early promise of modern logic—one might discover the secret of mathematics and human thought—and the pitfalls—one might get caught in contradiction. Indeed, many mathematicians and philosophers hesitated to pursue this new field precisely because of their fears that their work might turn out to be contradictory.

But the more adventurous went on. Because of the many brilliant aspects of Frege’s achievement, the fact that a basic premise was inconsistent came to be seen as a challenge, not a handicap. Repair the contradiction! Succeed where Frege failed! Modern logic did, indeed, overcome Frege’s mistake. It has become the language of mathematics and an essential tool of philosophy; it forms the foundation for computer science and has inspired the development of modern linguistics. For this reason, Frege’s achievements are honored by logicians and the story of his one, tragic mistake is memorialized under the title “Russell’s paradox.”

Would that you or I would make such a mistake as Frege’s!