CHAPTER 8 or HAMP

PERIODIC ORBITS FOR HAMILTONIAN SYSTEMS

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More about my recent results in the case $\mathbb{T}^n$ (sphere linking)? Define the radius of injectivity. Put HAMpartial here instead of in HAM? Add the proof of Theorem 33.5? Copy the precise statement of Arnold's conjecture.

We present here some results of existence and multiplicity of periodic orbits in Hamiltonian systems on cotangent bundles. Our main goal is to show the power, and relative simplicity of the method of decomposition by symplectic twist map as presented in Chapter HAM, which results into finite dimensional variational problems. Some of the results in this chapter have recently been improved upon by other authors. However, this was done at a high price, using hard analytic and topological method. Many of these, and other improvements could probably be obtained through the method presented here.

Bla bla bla....

42. Periodic Orbits In The Cotangent Of The n-Torus

We present here two results of existence and multiplicity of periodic orbits for Hamiltonian systems in $T^*\mathbb{T}^n$. The are easy corollaries of the Theorems of existence of multiple periodic orbits for symplectic twist maps proven in Chapter PSTM. The first one concerns a certain class of optical systems, the second one Hamiltonians that are quadratic nondegenerate outside of a bounded set.

A. Optical Hamiltonians

Assumption 42.1 (Uniform Opticity) $H(\boldsymbol{q}, \boldsymbol{p}, t) = H_t(\boldsymbol{z})$ is a twice differentiable function on $T^*\mathbb{T}^n \times \mathbb{R}$ (or $T^*M \times \mathbb{R}$, where $\tilde{M} = \mathbb{R}^n$) and satisfies the following:

(1) $\sup \left\|\nabla^2 H_t\right\| < K$

(2) The matrices $H_{\boldsymbol{pp}}(\boldsymbol{z}, t)$ are positive definite and its smallest eigenvalue are uniformly bounded below by $C > 0$.

Theorem 42.2 *Let $H(\boldsymbol{q}, \boldsymbol{p}, t)$ be a Hamiltonian function on $T^*\mathbb{T}^n \times \mathbb{R}$ satisfying Assumption 42.1. Then the time 1 map h^1 of the associated Hamiltonian flow has at least $n+1$ periodic orbits of type $\boldsymbol{m}, d$, for each prime $\boldsymbol{m}, d$, and 2^n when they are all non degenerate.*

Proof. We can decompose the time 1 map:

$$h^1 = h^1_{\frac{N-1}{N}} \circ \ldots \circ h^{\frac{k+1}{N}}_{\frac{k}{N}} \circ \ldots \circ h^{\frac{1}{N}}_0 .$$

and each of the maps $h^{\frac{k+1}{N}}_{\frac{k}{N}}$ is the time $\frac{1}{N}$ of the (extended) flow, starting at time $\frac{k}{N}$. Proposition HAMdecomponeof Chapter 4 shows that, for N big enough, such maps are symplectic twist maps. Moreover, we also noted in Chapter HAM, Remark HAMremthat these maps also satisfy the convexity condition . The result follows from Theorem STMPthesis. □

B. ASYMPTOTICALLY QUADRATIC HAMILTONIANS

We now turn to systems that are not necessarily optical, but satisfy a certain quadratic "boundary condition" which makes them completely integrable outside a compact set:

Theorem 42.3 *Let $H : T^*\mathbb{T}^n \times \mathbb{R} \to \mathbb{R}$ satisfy the following boundary condition:*

$$H(\boldsymbol{q}, \boldsymbol{p}, t) = \frac{1}{2}\langle A\boldsymbol{p}, \boldsymbol{p}\rangle + \boldsymbol{c}\cdot\boldsymbol{p}, \quad A^t = A, \det A \neq 0 \text{ when } \|\boldsymbol{p}\| \geq K. \tag{42.1}$$

Then h^1, the time–1 map of the Hamiltonian flow has at least $n+1$ distinct $\boldsymbol{m}, d$–orbits, and 2^n when they are all nondegenerate (i.e. generically). Furthermore, such an orbit lays entirely in the set $\|\boldsymbol{p}\| \leq K$ if and only if the rotation vector $\boldsymbol{m}/d$ belongs to the ellipsoid:

$$\mathcal{E} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \left\|A^{-1}(\boldsymbol{x} - \boldsymbol{c})\right\| \leq K\}.$$

Proof. The boundary condition (42.1) is Assumption 2 preceeding Theorem 39.5 , in which it is proven that the time ϵ of such Hamiltonians are twist maps. Hence, as remarked in Proposition HAMdecomptwo, the time 1 map can be decomposed into symplectic twist maps. To insure that these twist maps satisfy the conditions of Theorem STMPtquad, we go back to the proof of that proposition, and note that, instead of $G(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q}+\boldsymbol{p}, \boldsymbol{p})$, we can take $G(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q}+A\boldsymbol{p}+\boldsymbol{c}, \boldsymbol{p})$, the time 1 map of $H_0(\boldsymbol{q}, p) = \frac{1}{2}\langle A\boldsymbol{p}, \boldsymbol{p}\rangle + \boldsymbol{c}.\boldsymbol{p}$, obviously a symplectic twist map . Then, outside the set $\|\boldsymbol{p}\| \leq K$, the maps F_{2k}, F_{2k-1} of the decomposition are respectively the time 1 and the time $(\frac{1}{N} - 1)$ of the Hamiltonian flow associated to H_0, that is:

$$\begin{aligned} F_{2k}(\boldsymbol{q}, \boldsymbol{p}) &= (\boldsymbol{q} + A\boldsymbol{p} + \boldsymbol{c}, \boldsymbol{p}) \\ F_{2k-1}(\boldsymbol{q}, \boldsymbol{p}) &= (\boldsymbol{q} + \frac{1-N}{N}(A\boldsymbol{p} + \boldsymbol{c}), \boldsymbol{p}). \end{aligned}$$

These maps clearly satisfy the conditions of Theorem STMPtquad, which proves the existence of the advertised number of $\boldsymbol{m}, d$ orbits.

To localize these orbits, note that an orbit starting in $\|\boldsymbol{p}\| \geq K$ must stay there, and the map h^1 on such an orbit is just G. The rotation number of such an orbit is thus

$$(\boldsymbol{Q} - \boldsymbol{q}) = A\boldsymbol{p} + \boldsymbol{c}$$

from which we conclude that $\boldsymbol{m}/d$ is in the complement of $\mathcal{E}$. □

Remark 42.4 There is a distinction between periodic orbits of h^1 and periodic orbits of the Hamiltonian equations: for a general time dependent Hamiltonian flow, $(h^1)^n \neq h^n$, and hence an $\boldsymbol{m}, d$ periodic orbit for h^1 is not necessarily one for the O.D.E. (which should satisfy $h^{t+d}(\boldsymbol{z}) = h^t(\boldsymbol{z}) + (k\boldsymbol{m}, 0)$ for all $t \in [kd, (k+1)d), k \in \mathbb{Z}$). However, if H is periodic in time, of period 1, the equality $(h^1)^n = h^n$ does hold, and in this case the two notions coincide. In particular, this holds trivially for time independant Hamiltonians. Unfortunatly, these cases are degenerate in our setting, since $Dh^d(\boldsymbol{z})$ preserves the vector field X_H, which is thus an eigenvector with eigenvalue one. So in these cases, we can only claim the cuplength estimates for the number of periodic orbits for the Hamiltonian *flow* in either Theorem 42.2 or 42.3. We think that some further argument should yield, even in the time periodic case the sum of the betti number estimate for the number of flow periodic orbits, when the periodic orbits are nondegenerate *as orbits of the flow*: i.e., the only eigenvector of eigenvalue one for $Dh^d(\boldsymbol{z})$ is in the direction of the vector field X_H.

C. BIBLIOGRAPHY...

43. Periodic Orbits In General Cotangent Spaces

We now turn to the study of Hamiltonian systems in cotangent spaces of arbitrary compact manifolds. Our main result, which first appeared in Golé (1994) is:

Theorem 43.1 *Let (M, g) be a compact Riemannian manifold. Let $F : T^*M \to T^*M$ be the time 1 map of a time dependent Hamiltonian H on B^*M, where H is a C^2 function satisfying the boundary condition:*

$$H(\boldsymbol{q}, \boldsymbol{p}, t) = g(\boldsymbol{q})(\boldsymbol{p}, \boldsymbol{p}) \text{ for } \|\boldsymbol{p}\| \geq C.$$

where C is strictly smaller than the radius of injectivity ???have to define it somewhere???. Then F has $cl(M)$ distinct fixed points and $sb(M)$ if they are all non degenerate. Moreover, these fixed points lie inside the set $\{\|\boldsymbol{p}\| < C\}$ and can all be chosen to correspond to homotopically trivial closed orbits of the Hamiltonian flow.

THE DISCRETE VARIATIONAL SETTING

Define

$$B^*M = \{(\boldsymbol{q}, \boldsymbol{p}) \in T^*M \mid g(\boldsymbol{q})(\boldsymbol{p}, \boldsymbol{p}) = \|\boldsymbol{p}\|^2 \leq C^2 < R^2\},$$

where R is the radius of injectivity of (M, g). Let π denote the canonical projection $\pi : B^*M \to M$. Let F be as in Theorem 43.1. From Proposition HAMdecomptwoin Chapter HAM, we can decompose F into a product of symplectic twist maps :

$$F = F_{2N} \circ \ldots \circ F_1,$$

where F_{2k} restrained to the boundary ∂B^*M of B^*M is the time 1 map h_0^1 of the geodesic flow with Hamiltonian $H_0(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2}\|\boldsymbol{p}\|^2$. Likewise, F_{2k-1} is $h_0^{\frac{1-N}{N}}$ on ∂B^*M.

Let S_k be the generating function for the twist map F_k and $\psi_k = \psi_{F_k}$ the diffeomorphism $(q, p) \to (q, Q)$ induced by the twist condition on F_k. We can assume that ψ_k is defined on a neighborhood U of B^*M in T^*M. Let

(44.1)
$$O = \{\overline{\boldsymbol{q}} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{2N}) \in M^{2N} \,|(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) \in \psi_k(U) \text{ and } (\boldsymbol{q}_{2N}, \boldsymbol{q}_1) \in \psi_{2N}(U)\}$$

O is an open set in M^{2N}, containing a copy of M (the elements $\overline{\boldsymbol{q}}$ such that $\boldsymbol{q}_k = \boldsymbol{q}_1$, for all k).

Next, define :

(4.2)
$$W(\overline{\boldsymbol{q}}) = \sum_{k=0}^{2N} S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}),$$

where we have set $\boldsymbol{q}_{2N+1} = \boldsymbol{q}_1$. Choosing to work in some local coordinates around $\overline{\boldsymbol{q}} \in M^{2N}$, we let $\boldsymbol{p}_k = -\partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$ and $\boldsymbol{P}_k = \partial_2 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$. In other words, $(\boldsymbol{q}_k, \boldsymbol{p}_k) \in T^*_{\boldsymbol{q}_k} M$ is such that $\psi_k(\boldsymbol{q}_k, \boldsymbol{p}_k) = (\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$ and $(\boldsymbol{q}_{k+1}, \boldsymbol{P}_k) \in T^*_{\boldsymbol{q}_{k+1}} M$ is such that $F_k(\boldsymbol{q}_k, \boldsymbol{p}_k) = (\boldsymbol{q}_{k+1}, \boldsymbol{P}_k)$. We let the reader check that the following proofs can be written in coordinate free notation.

As in the case $M = \mathbb{T}^n$, we have:

Lemma 44.1 (Critical Action Principle) *The sequence $\overline{\boldsymbol{q}}$ of O is a critical point of W if and only if the sequence $\{(\boldsymbol{q}_k, \boldsymbol{p}_k)\}_{k\in\{1,\ldots,2N,1\}}$ is an orbit under the successive F_k's, that is if and only if $(\boldsymbol{q}_1, \boldsymbol{p}_1)$ is a fixed point for F.*

Proof. Because the twist maps are exact symplectic and using the definitions of $\boldsymbol{p}_k$, $\boldsymbol{P}_k$, we have:

(44.2)
$$\boldsymbol{P}_k d\boldsymbol{q}_{k+1} - \boldsymbol{p}_k d\boldsymbol{q}_k = dS_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}),$$

and hence
$$dW(\overline{\boldsymbol{q}}) = \sum_{k=1}^{2N} (\boldsymbol{P}_{k-1} - \boldsymbol{p}_k) d\boldsymbol{q}_k$$
which is null exactly when $\boldsymbol{P}_{k-1} = \boldsymbol{p}_k$, i.e. when $F_k(\boldsymbol{q}_{k-1}, \boldsymbol{p}_{k-1}) = (\boldsymbol{q}_k, \boldsymbol{p}_k)$. Now remember that we assumed that $\boldsymbol{q}_{2N+1} = \boldsymbol{q}_1$. □

Hence, to prove Theorem 43.1, we need to find enough critical points for W. As before, we will study the gradient flow of W (where the gradient will be given in terms of the metric g) and use the boundary condition to find an isolating block. The main difference with the previous situations on $T^*\mathbb{T}^n$ is that we cannot put W in the general framework of generating phases quadratic at infinity. Nonetheless, thanks to the boundary condition we imposed on the Hamiltonian, we are able to construct an isolating block and use Floer's theorem of continuation to get a grasp on the topology of the invariant set, and hence on the number of critical points.

45. Proof Of Theorem 43.1

THE ISOLATING BLOCK

In this subsection we prove that the set B defined as follows:

(45.1)
$$B = \{\overline{\boldsymbol{q}} \in O \mid \|\boldsymbol{p}_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})\| \leq C\}$$

is an isolating block for the gradient flow of W, where O is defined in (44.1) , C is as in the hypotheses of Theorem 43.1 and $\boldsymbol{p}_k = -\partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$. Note that when $\big\|\boldsymbol{p}_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})\big\| = C$,

$$\text{Dis}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) = |a_k|C \quad \text{where} \begin{cases} a_k &= 1 & \text{if } k \text{ is even} \\ a_k &= \frac{1-N}{N} & \text{if } k \text{ is odd} \end{cases} \tag{45.2}$$

Clearly B contains the constant sequences, a set homeomorphic to M.

Proposition 45.1 *B is an isolating block for the gradient flow of W.*

Proof. Suppose that the point $\overline{\boldsymbol{q}}$ of U is in the boundary of B. This means that $\|\boldsymbol{p}_k\| = C$ for at least one k. As noted in (45.2) , this means that $\text{Dis}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) = |a_k|C$ for some factor a_k only depending on the parity of k. We want to show that this distance increases either in positive or negative time along the gradient flow of W. This flow is given by:

$$\dot{\boldsymbol{q}}_k = A_k(\boldsymbol{P}_{k-1} - \boldsymbol{p}_k) = \nabla W_k(\overline{\boldsymbol{q}}) \tag{45.3}$$

where $A_k = A(\boldsymbol{q}_k)$ is the inverse of the matrix of coefficients of the metric g at the point $\boldsymbol{q}_k$. Remember that we have put the product metric on O, induced by its inclusion in M^{2N} (see Remark HAMgradon the definition of the gradient of a function).

We compute the derivative of the distance along the gradient flow at a boundary point of B, using Corollary HAMpartialand the fact that $h_0^{a_k}(\boldsymbol{q}_k, \boldsymbol{p}_k) = (\boldsymbol{q}_{k+1}, \boldsymbol{P}_k)$:

$$\begin{aligned} \frac{d}{dt}\text{Dis}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})\big|_{t=0} &= \partial_1 \text{Dis}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) \cdot \nabla W_k(\overline{\boldsymbol{q}}) \\ &\quad + \partial_2 \text{Dis}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) \cdot \nabla W_{k+1}(\overline{\boldsymbol{q}}) \\ &= sign(a_k)\frac{-\boldsymbol{p}_k}{\|\boldsymbol{p}_k\|} \cdot A_k(\boldsymbol{P}_{k-1} - \boldsymbol{p}_k) \\ &\quad + sign(a_k)\frac{\boldsymbol{P}_k}{\|\boldsymbol{P}_k\|} \cdot A_{k+1}(\boldsymbol{P}_k - \boldsymbol{p}_{k+1}) \end{aligned} \tag{45.4}$$

We now need a simple linear algebra lemma to treat this equation.

Lemma 45.1 *Let $\langle\, , \,\rangle$ denote a positive definite bilinear form in $\mathbb{R}^n$, and $\|.\|$ its corresponding norm. Suppose that $\boldsymbol{p}$ and $\boldsymbol{p}'$ are in $\mathbb{R}^n$,that $\|\boldsymbol{p}\| = C$ and that $\|\boldsymbol{p}'\| \leq C$. Then :*

$$\langle \boldsymbol{p}\ ,\ \boldsymbol{p}' - \boldsymbol{p}\ \rangle \leq 0.$$

Moreover, equality occurs if and only if $\boldsymbol{p}' = \boldsymbol{p}$.

Proof. From the positive definiteness of the metric, we get:

$$\langle\, \boldsymbol{p}' - \boldsymbol{p}, \boldsymbol{p}' - \boldsymbol{p}\, \rangle \geq 0,$$

with equality occuring if and only if $\boldsymbol{p}' = \boldsymbol{p}$. From this, we get:

$$2\langle\, \boldsymbol{p}, \boldsymbol{p}'\, \rangle \leq \langle\, \boldsymbol{p}', \boldsymbol{p}'\, \rangle + \langle\, \boldsymbol{p}, \boldsymbol{p}\, \rangle$$

Finally,

$$\langle\, (\boldsymbol{p}' - \boldsymbol{p}), \boldsymbol{p}\, \rangle = \langle\, \boldsymbol{p}', \boldsymbol{p}\, \rangle - \langle\, \boldsymbol{p}, \boldsymbol{p}\, \rangle \leq \frac{1}{2}(\langle\, \boldsymbol{p}', \boldsymbol{p}'\, \rangle - \langle\, \boldsymbol{p}, \boldsymbol{p}\, \rangle) \leq 0$$

with equality occuring if and only if $\boldsymbol{p}' = \boldsymbol{p}$. □

Applying Lemma 45.1 to each of the right hand side terms in (45.4) , we can deduce that $\frac{d}{dt}\mathrm{Dis}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$ is positive when k is pair, negative when k is odd. Indeed, because of the boundary condition in the hypothesis of the theorem, we have $\|\boldsymbol{P}_k\| = \|\boldsymbol{p}_k\|$ whenever $\|\boldsymbol{p}_k\| = C$: the boundary ∂B^*M is invariant under F and all the F_k's. On the other hand $\overline{\boldsymbol{q}} \in B \Rightarrow \|\boldsymbol{p}_l\| \leq C$ and $\|\boldsymbol{P}_l\| \leq C$, for all l, by invariance of B^*M. Finally, a_k is positive when k is even, negative when k is odd.

We have shown that the gradient flow exits B at all the points of ∂B except perhaps at the edges of ∂B. These edges are the sets of points $\overline{\boldsymbol{q}}$ such that more than one $\boldsymbol{p}_k$ has norm C. The problem at these edges occurs when k is in an interval $\{l, \ldots, m\}$ such that, for all j in this interval, $\|\boldsymbol{p}_j\| = C = \|\boldsymbol{P}_j\|$ and $\nabla W_j(\overline{\boldsymbol{q}}) = 0$.

It is now crucial to note that $\{l, \ldots, m\}$ can not cover all of $\{0, \ldots, 2N\}$: this would mean that $\overline{\boldsymbol{q}}$ is a critical point corresponding to a fixed point of h_0^1 in ∂B^*M. But such a fixed point is forbidden by our choice of C: orbits of our Hamiltonian on the set $\|\boldsymbol{p}\| = C$ are geodesics, but geodesics in that energy level can not be fixed loops since $C > 0$, and they can not close up in time one either since C is less than the injectivity radius.

We now let $k = m$ in (45.4) and see that the flow must definitely escape the set B at $\overline{\boldsymbol{q}}$ in either positive or negative time, from the m^{th} face of B. □

Remark 45.2 If the Hamiltonian considered is optical and we decompose its time 1 map into a product of N twist maps as in HAMdecompone, all the F_k's coincide with $h_0^{\frac{1}{N}}$ on the boundary of B^*M. In that case, all the a_k's in the above proof are positive, and B is a repeller block in this case.

END OF PROOF OF THEOREM 43.1

To finish the proof of Theorem 43.1 we use Floer's theorem TOPOfloerthmof continuation of normally hyperbolic invariant sets. We consider the family F_λ of time 1 maps of the Hamiltonians:

$$H_\lambda = (1-\lambda)H_0 + \lambda H.$$

Corresponding to this is a family of gradient flows ζ_λ^t, solution of

$$\frac{d}{dt}\overline{\boldsymbol{q}} = \nabla W_\lambda(\overline{\boldsymbol{q}}),$$

where W_λ is the discrete action corresponding to the decomposition in symplectic twist maps of the map F_λ. We take care that this decomposition has the same number of steps $2N$ for each λ. The manifold on which we consider these (local) flows is O, which is an open neighborhood of B in M^{2N}. Each of the F_λ satisfies the hypothesis of Theorem 43.1, and thus Proposition 45.1 applies to ζ_λ^t for all λ in $[0, 1]$: B is an isolating block for each one of these flows. Hence the maximum invariant sets G_λ for the flows ζ_λ^t in B are related by continuation. The part of Floer's Theorem that we need to check is that G_0 is a normally hyperbolic invariant manifold for ζ_0^t.

Lemma 45.2 *Let $G_0 = \{\overline{\boldsymbol{q}} \in B \mid \boldsymbol{q}_k = \boldsymbol{q}_1, \forall k\}$. Then G_0 is a normally hyperbolic invariant set for ζ_0^t. G_0 is a retract of O and it is the maximal invariant set in B.*

Proof. The only critical points for W_0 in B are the points of G_0 which correspond to restpoints of the geodesic flow, *i.e.* the zero section. Indeed, critical points of W_0 in B corresponds to periodic points of period 1 for the geodesic flow in B^*M. Our definition of that sets precludes nontrivial periodic geodesics in B^*M. We now show thatthe maximum invariant set for ζ_0^t in B is included in G_0. Since ζ_0^t is a gradient flow, such an invariant set is formed by critical points and connecting orbits between them. The only critical points of W_0 in B are the points of G_0. If there were a connecting orbit entirely in B, it would have to connect two points in G_0, which is absurd since $W_0 \equiv 0$ on G_0, whereas W_0 should increase along non constant orbits.

G_0 is a retract of M^{2N} under the composition of the maps:

$$\overline{\boldsymbol{q}} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{2N}) \rightarrow \boldsymbol{q}_1 \rightarrow (\boldsymbol{q}_1, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_1) = \alpha(\overline{\boldsymbol{q}})$$

which is obviously continuous and fixes the points of G_0.

It remains to show that G_0 is normally hyperbolic. Since $G_0 \cong M$ is an n-dimensional manifold made of critical points, saying that it is normally hyperbolic is equivalent to saying that $ker\nabla^2 W_0(\overline{\boldsymbol{q}})$ has dimension n: indeed, if it is the case, the only possible vectors in this kernel must be tangent to G_0, and thus he differential of the flow is nondegenerate on the normal space to TG_0. In the present situation, the second variation formula of Lemma 31.2says that the 1-eigenspace of Dh_0^1 is isomorphic to the kernel of $\nabla^2 W_0$. Hence it is enough to check that at a point $(\boldsymbol{q}_1, 0) \in B^*M$ corresponding to $\overline{\boldsymbol{q}}$, 1 is an eigenvalue of multiplicity exactly n for $Dh_0^1(\boldsymbol{q}_1, 0)$. Let us compute $Dh_0^1(\boldsymbol{q}_1, 0)$ in local coordinates. It is the solution at time 1 of the linearized (or variation) equation:

$$\dot{U} = J\nabla^2 H_0(\boldsymbol{q}_1, 0)U$$

along the constant solution $(q(t), p(t)) = (\boldsymbol{q}_1, 0)$, where J denotes the usual symplectic matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. An operator solution for the above equation is given by $exp\left(tJ\nabla^2 H_0(\boldsymbol{q}_1, 0)\right)$ On the other hand:

$$\nabla^2 H_0(\boldsymbol{q}_1, 0) = \begin{pmatrix} 0 & 0 \\ 0 & A(\boldsymbol{q}_1) \end{pmatrix}$$

which we computed from $H_0(q,p) = A(q)p.p$, the zero terms appearing at $p = 0$ because they are either quadratic or linear in p. From this,

$$Dh_0^1(\boldsymbol{q}_1, 0) = exp\left(J\nabla^2 H_0(\boldsymbol{q}_1, 0)\right) = \begin{pmatrix} I & A(\boldsymbol{q}_1) \\ 0 & I \end{pmatrix}$$

is easily derived. This matrix has exactly n independent eigenvectors of eigenvalue 1 (it has in fact no other eigenvector). Hence, from Lemma 31.2, $\nabla^2 W(\overline{\boldsymbol{q}})$ has exactly n vectors with eigenvalue 0, as was to be shown. □

We now conclude the proof of Theorem 43.1. We have proved that the gradient flow ζ^t, has an invariant set G_1 with $H^*(M) \hookrightarrow H^*(G_1)$. From this we get in particular:

$$cl(G_1) \geq cl(M) \text{ and } sb(G_1) \geq sb(M).$$

Theorem 50.2 tells us that ζ^t must have at least $cl(G_1)$ rest points in the set G_1, and $sb(G_1)$ if all rest points are nondegenerate. But Lemma 31.2tells us that nondegeneracy for $\nabla^2 W$ at a critical point is the same thing as nondegeneracy of a fixed point for F (no eigenvector of eigenvalue 1). This proves the existence of the advertised number of fixed points of the map F. In the following section, we will see that all these fixed points of the time 1 map correspond to periodic orbits may be chosen to be homotopically trivial. This concludes the proof of Theorem 43.1. □

FREE HOMOTOPY CLASSES

Since each F_k is close (or equal) to $h_0^{a_k}$ for some positive or negative a_k, we have: $\boldsymbol{q}$ is in the set $\psi_k(B^*_{\boldsymbol{q}}M)$ and, since $B^*_{\boldsymbol{q}}M \to \psi_k(B^*M)$ is a diffeomorphism, we can define a path $c_k(q,Q)$ between q and a point Q of $\psi_k(B^*_{\boldsymbol{q}}M)$ by taking the image by ψ_k of the oriented line segment between $\psi_k^{-1}(q)$ and $\psi_k^{-1}(Q)$ in $B^*_{\boldsymbol{q}}M$. In the case where $F_k = h_0^1$, this amounts to taking the unique geodesic between q and Q in $\psi_k(B^*_{\boldsymbol{q}}M)$.

If we look for periodic orbits of period d and of a given homotopy type, we decompose F^d into $2Nd$ twist maps, by decomposing F into $2N$. Analogously to (4.1), we define :

$$O_d = \{\overline{q} = (q_1, \dots, q_{2Nd}) \in M^{2Nd} \,|(q_k, q_{k+1}) \in \psi_k(U) \text{ and } (q_{2Nd}, q_1) \in \psi_{2Nd}(U)\},$$

remarking that the ψ_k's here correspond to the decomposition of F^d into $2Nd$ steps (U is as before a neighborhood of B^*M).

To each element $\overline{\boldsymbol{q}}$ in O_d, we can associate a closed curve $c(\overline{\boldsymbol{q}})$, made by joining up each pair (q_k, q_{k+1}) with the curve $c_k(q_k, q_{k+1})$ uniquely defined as above. This loop $c(\overline{\boldsymbol{q}})$ is piecewise differentiable and it depends continuously on $\overline{\boldsymbol{q}}$, and so do its derivatives (left and right). In the case of the decomposition of h_0^1 , taking F_k=h_0^1, this is exactly the construction of the broken geodesics (see Section 38.0). Now any closed curve in M belongs to a free homotopy class m.[10] To any d periodic point for F, we can associate a sequence $\overline{\boldsymbol{q}}(x) \in O_d$ of q coordinates of the orbit of this point under the successive F_k's in the decomposition of F^d.

Definition 45.3 Let $\boldsymbol{z}$ be a periodic point of period d for F. Let $\overline{\boldsymbol{q}}$ be the sequence in O_d corresponding to x. We say that x is an (m, d) point if $c(\overline{\boldsymbol{q}}(x))$ is in the free homotopy class m.

This definition has the advantage to make sense for any map F of T^*M which can be decomposed into the product of symplectic twist maps . If F is also the time 1 map of a Hamiltonian, it agrees with the obvious definition:

Proposition 45.4 *If $\boldsymbol{z}$ is an $\boldsymbol{m}, d$ periodic orbit, then the projection $\pi(\boldsymbol{z}(t)), t \in [0, d]$ of the orbit of $\boldsymbol{z}$ under the Hamiltonian flow is a closed curve in the free homotopy class $\boldsymbol{m}$.*

Proof. Left as an exercise (*Hint.* Use the geodesic flow to construct the homotopy between $c(\overline{\boldsymbol{q}}(\boldsymbol{z}))$ and $\pi(\boldsymbol{z}(t))$.)

Let

$$O_{m,d} = \{\overline{\boldsymbol{q}} \in O \mid c(\overline{\boldsymbol{q}}) \in m\} \qquad (45.5)$$

Since $c(\overline{\boldsymbol{q}})$ depends continuously on $\overline{\boldsymbol{q}} \in O$, $O_{m,d}$ is a connected component of O. The reader who wants to make sure that, in the proof of Theorem 43.1, the orbits found are homotopically trivial, can check that the

[10] We remind the reader that free homotopy classes of loops differ from elements of $\pi_1(M)$ in that no base point is kept fixed under the homotopies. As a result, free homotopy classes can be seen as conjugacy classes in $\pi_1(M)$, and thus can not be endowed with a natural algebraic structure. Two elements of a free class give the same element in $H_1(M)$. Hence free homotopy classes form a set smaller than $\pi_1(M)$, bigger than $H_1(M)$. All these sets coincide if $\pi_1(M)$ is abelian.

proof we gave in last section works identically when one replace the space O, by its connected component $O_{e,1}$, where e is the homotopy class of the trivial curve. Another place where one uses this decomposition of O in different homotopy components is the following:

Theorem 45.5 *Let (M,g) be a Riemannian manifold of negative curvature and H be as in Theorem 1. If $\gamma_{\boldsymbol{m}}$ denotes the (unique) closed geodesic of free homotopy class $\boldsymbol{m}$, F has at least 2 (m,d) orbits in B^*M when length$(\gamma_{\boldsymbol{m}}) < dC$.*

The proof of Theorem 45.5 (see Golé (1994), Theorem 2) has the same broad outline as that of Theorem 1. We work in $O_{\boldsymbol{m},d}$ instead of O. The normally hyperbolic invariant set that we continue to in this setting is given by the set G_0 of critical sequences corresponding to the orbits under the $h_0^{a_k}$'s of the points on $\gamma_{\boldsymbol{m}}$. The normal hyperbolicity of G_0 derives this time from the hyperbolicity of the geodesic flow in negative curvature.

??? Add the proof in? ???

46. Linking Of Spheres: Toward A Generalization Of The Theorem Of Poincaré And Birkhoff

As stated in the introduction , Arnold conjectured in 1965 a generalization of the Theorem of Poincaré-Birkhoff for Hamiltonian maps of $\mathbb{T}^n \times \mathbb{B}^n$ (where $\mathbb{B}^n$ is the closed ball in $\mathbb{R}^n$).

Arnold's Linking of Spheres Conjecture

Generalized Arnold Conjecture Let M be a compact manifold, and F be a Hamiltonian map of a ball bundle B^*M in T^*M. Suppose that each sphere ∂B_q^*M links with its image by F in ∂B^*M. Then F has at least $cl(M)$ distinct fixed points, and at least $sb(M)$ if they are nondegenerate.

In Banyaga & Golé (???)(see also Golé (1994)), we proved the simple case:

Theorem 46.1 *Let F be a symplectic twist map of B^*M which links spheres on the boundary ∂B^*M. Then F satisfies the generalized Arnold Conjecture.*

Proof. The proof of this theorem is trivial once one understands the meaning of the linking condition. If one looks at the Poincaré-Birkhoff situation, an easy equivalent condition to the boundary twist condition (points on the two boundary components go in opposite directions for some lift of F) is that a vertical fiber $\{x = x_0\}$ and its image by F should have a nonzero algebraic intersection number (*i.e.* the number of intersections counted with orientation). Let us take this for the moment as a working definition of the linking of spheres in the general case:

Definition 46.2 (Boundary Twist: version 1) We say that a map $F : B^*M \to B^*M$ satisfies the boundary twist condition if each fiber $\Delta_{q_0} = \pi^{-1}(\boldsymbol{q}_0)$ intersects its image by F with a nonzero algebraic intersection number

We will see later on (for the reader who is comfortable with a little algebraic topology) that this intersection number condition is equivalent to linking of the boundary spheres as is usually defined in algebraic topology (and was probably meant by Arnold). The importance of this is that the boundary twist condition is indeed a topological condition on the action of the map *on the boundary*.

If F is a symplectic twist map, a fiber $\Delta_{\boldsymbol{q}}$ and its image under F may intersect at most once. Hence the boundary twist condition means in this case that all the fibers intersect their image *exactly once*. Fixed points of F correspond to critical points of $\boldsymbol{q} \to S(\boldsymbol{q}, \boldsymbol{q})$. This function is well defined since, by what preceeds, the diagonal in $M \times M$ is in the image of B^*M by the embedding ψ_F. Hence F has as many fixed points as the function $\boldsymbol{q} \to S(\boldsymbol{q}, \boldsymbol{q})$ has critical points on M. Morse and Lyusternick-Schnirelman's theories give the advertised estimates. □

We now show that, in the case considered by Arnold, our working definition of boundary twist is indeed equivalent to the classical one of algebraic topology. We first remind the reader of the classical definition of linking of spheres. Let $\Delta_{\boldsymbol{q}}$ be a fiber of B^*M as before. Then $\partial\Delta_{\boldsymbol{q}}$ is an n dimensional sphere. It make sense to talk about its linking with its image $F(\partial\Delta_{\boldsymbol{q}})$ in $\partial B^*\tilde{\mathbb{T}}^n$: the latter set has dimension $2n-1$ and the dimensions of the spheres add up to $2n-2$. The linking number $F(\partial\Delta_{\boldsymbol{q}})$ with $\partial\Delta_{\boldsymbol{q}}$ is given by the class $[F(\partial\Delta_{\boldsymbol{q}})] \in H_{n-1}(\partial B^*\tilde{\mathbb{T}}^n \backslash \partial\Delta_{\boldsymbol{q}})$ More precisely, we have:

$$(46.1) \qquad \begin{aligned} H_{n-1}(\partial B^*\tilde{\mathbb{T}}^n \backslash \partial\Delta_{\boldsymbol{q}}) &\cong H_{n-1}\left(\mathbb{S}^{n-1} \times (\mathbb{R}^n - \{0\})\right) \\ &\overset{\text{Kunneth}}{\cong} H_{n-1}(\mathbb{S}^{n-1}) \oplus H_{n-1}(\mathbb{R}^n - \{0\}) \end{aligned}.$$

Thus, taking $\partial\Delta_{\boldsymbol{q}}$ from $\partial B^*\tilde{\mathbb{T}}^n$ creates a new generator in the $(n-1)$st homology, i.e. the generator b of $H_{n-1}(\mathbb{R}^n - \{0\})$.

The *linking number* of the spheres $F(\partial\Delta_{\boldsymbol{q}})$ and $\partial\Delta_{\boldsymbol{q}}$ is given by the $H_{n-1}(\mathbb{R}^n - \{0\}) \cong \mathbb{R}$ coefficient in the decomposition of the homology class $[F(\partial\Delta_{\boldsymbol{q}})]$ in the direct sum in (46.1) . If the linking number is nonzero, we say that the spheres $\partial\Delta_{\boldsymbol{q}}$ and its image by F *link*.

Definition (Boundary Twist: Version 2) We will say that the map F satisfies the *boundary twist condition* if for all $\boldsymbol{q} \in \tilde{\mathbb{T}}^n$ these spheres link in $\partial B^*\tilde{\mathbb{T}}^n$.

Lemma 46.3 *If F is the lift of a diffeomorphism of $B^*\mathbb{T}^n = \mathbb{T}^n \times B^n$, the two definitions of the boundary twist condition are equivalent. More precisely, the algebraic intersection number $\#(\Delta_{\boldsymbol{q}} \cap F(\Delta_{\boldsymbol{q}}))$ and the linking number of the spheres $\partial\Delta_{\boldsymbol{q}}$ and $F(\partial\Delta_{\boldsymbol{q}})$ are equal.*

Proof. We complete (46.1) into the following commutative diagram:

$$\begin{array}{ccc} H_{n-1}(\partial B^*\tilde{\mathbb{T}}^n \backslash \partial\Delta_{\boldsymbol{q}}) & \cong & H_{n-1}(\mathbb{R}^n - \{0\}) \oplus H_{n-1}(\mathbb{S}^n) \\ \downarrow i_* & & \downarrow j_* \\ H_{n-1}(B^*\tilde{\mathbb{T}}^n \backslash \Delta_{\boldsymbol{q}}) & \cong & H_{n-1}\left((\mathbb{R}^n - \{0\} \times B^n)\right) \end{array}$$

where i, j are inclusion maps. It is clear that $j_* b$ generates

$$H_{n-1}\left((\mathbb{R}^n - \{0\}) \times B^n\right) \cong H_{n-1}\left((\mathbb{R}^n - \{0\}) \times \mathbb{R}^n\right).$$

The last group measures the (usual) linking number of a sphere with the fiber Δ_q in $B^*\tilde{\mathbb{T}}^n \cong \mathbb{R}^{2n}$. But it is well known that such a number is the intersection number of any ball bounded by the sphere with the fiber Δ_q, counted with orientation. □

???more about my recent results in the case $\mathbb{T}^n$???

Theorem HAMPthmfp is 43.1, Theorem HAMPthmhyp is 45.5