CHAPTER 5 or STMP

PERIODIC ORBITS FOR SYMPLECTIC TWIST MAPS OF T*T^n

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29. Presentation Of The Results

Rewrite the ghost tori section (too silly!). Points to be made: parallele with Floer homology, possible dynamic use (2 dim case and more). The proof of Corollary 33.2 can be made using Conley theory only: do that if I get rid of Morse theory in TOPO

In this Chapter, we give some results on existence and multiplicity of periodic orbits of different rotation vectors for symplectic twist map of $T^*\mathbb{T}^n$. The introduction of more refined topological tools yield an improvement on the results of Golé (1989)(see also Golé (1991)).

Similarly to the case $n=1$, a point $(\boldsymbol{q},\boldsymbol{p}) \in \mathbb{R}2n$ is called a $\boldsymbol{m}, d$*–periodic point* for the lift F of a map f of $T^*\mathbb{T}^n$ if

$$F^d(\boldsymbol{q},\boldsymbol{p}) = (\boldsymbol{q}+\boldsymbol{m},\boldsymbol{p})$$

where $\boldsymbol{m} \in \mathbb{Z}^n$ and $d \in \mathbb{Z}^+$. The rational vector $\frac{\boldsymbol{m}}{d}$ is called the *rotation vector* of the orbit of $(\boldsymbol{q},\boldsymbol{p})$. In general, the rotation vector (when it exists) of a sequence $\{\boldsymbol{q}_k\}_{k\in\mathbb{Z}} \in (\mathbb{R}^n)^{\mathbb{Z}}$ is given by the limit: $\rho(\overline{\boldsymbol{q}}) = \lim_{k\to\pm\infty} \boldsymbol{q}_k$.

The maps that we consider here satisfy either one of the following two assumptions: $F = F_N \circ \ldots \circ F_1$ is the product of lifts of symplectic twist maps of $T^*\mathbb{T}^n$, with generating functions S_k such that either the following convexity or asymptotic linearity conditions:

Convexity There is a positive real a such that:

$$(29.1)\ (a) \qquad \langle \partial_{12} S_k(\boldsymbol{q},\boldsymbol{Q}).\boldsymbol{v},\boldsymbol{v}\rangle \le -a\left\|\boldsymbol{v}\right\|^2, \quad \forall \boldsymbol{q},\boldsymbol{Q},\boldsymbol{v} \in \mathbb{R}^n, k \in \{1,\ldots,N\}.$$

Equivalently:

$$(29.1)\ (b) \qquad F_k(\boldsymbol{q},\boldsymbol{p}) = (\boldsymbol{Q},\boldsymbol{P}) \quad \text{and} \quad \left\langle \left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{p}}\right)^{-1} \boldsymbol{v},\boldsymbol{v}\right\rangle \ge a\left\|\boldsymbol{v}\right\|^2, \quad \forall \boldsymbol{v} \in \mathbb{R}^n.$$

uniformly in $(\boldsymbol{q},\boldsymbol{p})$.

Asymptotic Linearity

$$S_k(\boldsymbol{q},\boldsymbol{Q}) = \frac{1}{2}\langle A_k(\boldsymbol{Q}-\boldsymbol{q}), (\boldsymbol{Q}-\boldsymbol{q})\rangle + R_k(\boldsymbol{q},\boldsymbol{Q})$$

with:

(29.2) (a)
$$A_k = A_k^t, \ \det A_k \neq 0$$

(29.2) (b)
$$\det \sum_1^N A_k^{-1} \neq 0$$

(29.2) (c)
$$\lim_{\|\boldsymbol{Q}-\boldsymbol{q}\|\to\infty} \frac{\nabla R_k(\boldsymbol{q},\boldsymbol{Q})}{\|\boldsymbol{Q}-\boldsymbol{q}\|} = 0.$$

Equivalently:

$$F_k(\boldsymbol{q},\boldsymbol{Q}) = (\boldsymbol{q} + A_k^{-1}\boldsymbol{p} + \Theta(\boldsymbol{q},\boldsymbol{p}), \ \boldsymbol{p} + \Upsilon(\boldsymbol{q},\boldsymbol{p}))$$

with (29.2) (a) and (b) holding for A_k and:

(29.2) (c')
$$\lim_{\|\boldsymbol{p}\|\to\infty} \frac{\Theta(\boldsymbol{q},\boldsymbol{p})}{\|\boldsymbol{p}\|} = \lim_{\|\boldsymbol{p}\|\to\infty} \frac{\Upsilon(\boldsymbol{q},\boldsymbol{p})}{\|\boldsymbol{p}\|} = 0$$

Theorem 29.1 *Let $F = F_N \circ \ldots \circ F_1$ be a finite composition of symplectic twist maps F_k of $T^*\mathbb{T}^n$ satisfying either the convexity condition (29.1) or the asymptotic condition (29.2) . Then, for each relatively prime $(\boldsymbol{m}, d) \in \mathbb{Z}^n \times \mathbb{Z}$, F has at least $n+1$ periodic orbits of type $\boldsymbol{m}, d$. It has at least 2^n of them when they are all non–degenerate.*

The proof of this theorem appeared in several pieces: the existence in the convex case was given by Kook & Meiss (1989). Their proof of multiplicity was corrected by the author in Golé (1994). The proof of the theorem with the asymptotic condition is the center of the author's thesis Golé (1989)(see also Golé (1991)). The proof we present here is also more unified, and hopefully simpler. It also improves on our previous results where, in certain cases, we could not garantee the existence of more than 2^{n-1} periodic orbits.

Comments on Conditions (29.1) and (29.2) . In Chapter STM, Proposition 26.5, we derived $\frac{\partial Q}{\partial p}(\boldsymbol{q},\boldsymbol{p}) = -\left(\partial_{12}S(\boldsymbol{q},\boldsymbol{Q})\right)^{-1}$, by implicit differentiation of $\boldsymbol{p} = -\partial_1 S(\boldsymbol{q},\boldsymbol{Q})$. The convexity condition (29.1) (a) thus translates to (29.1) (b). Note that (29.1) (b) means that F has bounded, positive definite twist. MacKay & al. (1989) imposed this condition on their definition of symplectic twist maps, a terminology that we have taken from them. Remember that Proposition 26.4in Chapter STM shows that the bounded twist condition (29.2) implies the global twist condition.

As for Condition (29.2) we stress that each A_k is *not necessarily positive definite*, but only a nondegenerate symmetric matrix. This is what Hermann (1990)called the *indefinite case*. If we set $R_k = 0$ in S_k, we obtain a quadratic generating function for a linear symplectic twist map $L_k(\boldsymbol{q},\boldsymbol{p}) = (\boldsymbol{q} + A_k^{-1}\boldsymbol{p}, \boldsymbol{p})$. Thus, if $L = L_N \circ \ldots \circ L_1$, condition (29.2) implies that

(29.3)
$$L(\boldsymbol{q},\boldsymbol{p}) = (\boldsymbol{q} + A\boldsymbol{p}, \boldsymbol{p}) \quad \text{with} \quad A = \sum_{k=1}^{dN} A_k^{-1}$$

is a symplectic twist map. Hence Condition (29.2) can be expressed as saying that F is asymptotically linear (and asymptotically completely integrable), in that it is close to L at ∞: (29.2) (c') shows that

$$\lim_{\|\boldsymbol{p}\|\to\infty} \frac{\|F(\boldsymbol{q},\boldsymbol{p}) - L(\boldsymbol{q},\boldsymbol{p})\|}{\|\boldsymbol{p}\|} = 0.$$

We leave it to the reader to show that the generating function and map conditions in (29.2) are indeed equivalent.

Example 29.2 The generalized standard map satisfies both conditions (29.1) and (29.2)

Outline of the proof. In the convex case, we start by finding a minimum for a discrete action function W, sum of generating functions. The convexity condition, as in the classical calculus of variation gives us coercion on W, which implies the existence of the minimum. The multiplicity is given by Morse theory on an adequately chosen sublevel set $\{W \leq C\}$.

The case with the asymptotic condition is a relatively easy consequence of Proposition 52.8: we find that the action function W on the appropriate quotient space of the space of sequences is indeed quadratic at infinity as required by that Proposition.

30. Finite Dimensional Variational Setting

Let $F = F_N \circ \ldots \circ F_1$ where each F_k is the lift of a symplectic twist map with generating function S_k. The critical action principle in Chapter STM tells us that finding orbits of F can be done by finding solutions of:

$$\partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) + \partial_2 S_{k-1}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k) = 0 \tag{30.1}$$

The appropriate space of sequences in which to look for solutions of (30.1) corresponding to $\boldsymbol{m}, d$–points of F is:

$$\overline{\mathbf{X}} = \{\overline{\boldsymbol{q}} \in (\mathrm{I\!R}^n)^{\mathbb{Z}} \mid \boldsymbol{q}_{k+dN} = \boldsymbol{q}_k + \boldsymbol{m}\}$$

which is isomorphic to $(\mathrm{I\!R}^n)^{dN}$: the terms $(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{dN})$ determine a whole sequence in $\overline{\mathbf{X}}$, and we will use them as a coordinate system for this space. Finding a sequence satisfying (30.1) in $\overline{\mathbf{X}}$, is equivalent to finding $\overline{\boldsymbol{q}} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{dN})$ which is a critical point for the function:

$$W(\overline{\boldsymbol{q}}) = \sum_{k=1}^{dN} S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}),$$

in which we set $\boldsymbol{q}_{dN+1} = \boldsymbol{q}_1$. In fact, the proof of the critical action principle (see Proposition 24.1and also Corollary 5.2) reduces in this case to the suggestive formula:

$$dW(\overline{\boldsymbol{q}}) = \sum_{k=1}^{dN} (\boldsymbol{P}_{k-1} - \boldsymbol{p}_k) d\boldsymbol{q}_k. \tag{30.2}$$

The search for critical points of W will be made by studying the gradient flow solution of

$$\frac{d\overline{\boldsymbol{q}}(t)}{dt} = -\nabla W(\overline{\boldsymbol{q}}(t))$$

where t is an artificial time variable. Written in components, this equation is the differential equation:

$$\dot{\boldsymbol{q}}_k = -\partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) - \partial_2 S_{k-1}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k)$$

which for C^2 functions S_k's defines a local flow φ^t on $\overline{\mathbf{X}}$. This flow will certainly be defined for all $t \in \mathbb{R}$ whenever the second derivatives of the S_k's are bounded: the vector field $-\nabla W$ is then globally Lipschitz.

We need to complicate matters some more. First, notice that $\overline{\mathbf{X}}$ has trivial topology, so we should take advantage of the periodicity of W. Formally, this can be done by remarking that W is invariant under the diagonal $\mathbb{Z}^n$ action: $W \circ \tau_{\boldsymbol{n}} = W,\ \boldsymbol{n} \in \mathbb{Z}^n$ where

$$\tau_{\boldsymbol{n}}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{dN}) = (\boldsymbol{q}_1 + \boldsymbol{n}, \ldots, \boldsymbol{q}_{dN} + \boldsymbol{n}).$$

Hence W induces a function on the quotient $\overline{\mathbf{X}}/\mathbb{Z}^n$. This operation takes in account the fact that the maps F and F_k are all lifts of maps of $T^*\mathbb{T}^n$. Without this condition it is easy to find maps of $\mathbb{R}2n$ without $\boldsymbol{m}, d$-orbits,*eg.* $(\boldsymbol{q}, \boldsymbol{p}) \mapsto (\boldsymbol{q}, \boldsymbol{p} + \boldsymbol{a})$.

But we go one step further. We are not satisfied with finding distinct $\boldsymbol{m}, d$–points, but we want to make sure that different critical points of our function W correspond in fact to different $\boldsymbol{m}, d$*–orbits* of F. To this effect, we note that W is also invariant under the N^{th} iterate σ^N of the shift map:

$$(\sigma\overline{\boldsymbol{q}})_k = \boldsymbol{q}_{k+1}.$$

This is because $S_{k+N} = S_k$, and thus σ^N permutes circularly the terms of W. Hence we can define W succesively on the quotients:

$$\overline{\mathbf{X}} = \overline{\mathbf{X}}/\tau = \overline{\mathbf{X}}/\mathbb{Z}^n \quad \text{and}$$
$$\mathbf{X} = \overline{\mathbf{X}}/\sigma^N = \overline{\mathbf{X}}/(\mathbb{Z}^n \times \mathbb{Z})$$

of $\overline{\mathbf{X}}$ by the actions of $\tau_{\boldsymbol{n}},\ \boldsymbol{n} \in \mathbb{Z}^n$ and σ^N. Since the action of σ^N on critical sequences corresponds to the action of F on points of $T^*\mathbb{T}^n$, distinct critical points of W on $\mathbf{X}$ correspond to distinct orbits of F.

The following lemma, due to Bernstein & Katok (1987), describes the topology of the problem:

Lemma 30.1 *The quotient maps:* $\overline{\mathbf{X}} \to \overline{\mathbf{X}}$ *and* $\overline{\mathbf{X}} \to \mathbf{X}$ *are covering maps , and thus so is* $\overline{\mathbf{X}} \to \mathbf{X}$. *The space* $\overline{\mathbf{X}}$ *is homeomorphic to* $\mathbb{T}^n \times (\mathbb{R}^n)^{dN-1}$, *whereas* $\mathbf{X}$ *is a (not always trivial) fiber bundle with base* $\mathbb{T}^n$ *and fiber* $(\mathbb{R}^n)^{dN-1}$.

Proof. We make the change of variables:

$$\boldsymbol{q} = \frac{1}{dN}\sum_1^{dN} \boldsymbol{q}_k$$
$$\boldsymbol{v}_k = \boldsymbol{q}_{k+1} - \boldsymbol{q}_k - \boldsymbol{m}/dN, \quad k \in \{1, \ldots, dN-1\}$$

and think of $\boldsymbol{q}$ as the base coordinate and $\boldsymbol{v}$ as the fiber. In these coordinates:

$$\tau_{\boldsymbol{n}}(\boldsymbol{q}, \boldsymbol{v}) = (\boldsymbol{q} + \boldsymbol{n}, \boldsymbol{v})$$
$$\sigma(\boldsymbol{q}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{dN-1}) = \left(\boldsymbol{q} + \frac{\boldsymbol{m}}{dN}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{dN-1}, -\sum_{j=1}^{dN-1} \boldsymbol{v}_j\right)$$
$$\sigma^{dN}(\boldsymbol{q}, \boldsymbol{v}) = (\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{v})$$

(the reader should verify this...) From the first equality, we get:

$$\overline{\mathbf{X}} \overset{\text{def}}{=} \overline{\mathbf{X}}/\mathbb{Z}^n \simeq \mathbb{T}^n \times (\mathbb{R}^n)^{dN-1}.$$

and σ^N induces a d–periodic, fixed point free diffeomorphism on $\overline{\mathbf{X}}$, and thus taking the quotient of $\overline{\mathbf{X}}$ by σ^N gives again a covering map. Finally, these coordinates show that $\mathbf{X} = \overline{\mathbf{X}}/\sigma^N$ is a fiber bundle over $(\mathbb{R}^n/\mathbb{Z}^n)/\frac{m}{d}\mathbb{Z} \simeq \mathbb{T}^n$. □

31. Second Variation

In this section, we show how the second derivative of W can be used to decide if a periodic orbit is nondegenerate or not.

Definition 31.1 A periodic point $\boldsymbol{z}$ of period d for a symplectic twist map F is called *nondegenerate* if $DF^d_{\boldsymbol{z}}$ has no eigenvalue 1.

Suppose $F = F_N \circ \ldots \circ F_1$ where each F_k is a symplectic twist map and let W be defined as before.

Lemma 31.2 *An $\boldsymbol{m}, d$ periodic point is nondegenerate for F if and only if the critical point of W to which it corresponds is nondegenerate.*

Proof. Suppose that $(\boldsymbol{q}_1, \boldsymbol{p}_1) = \boldsymbol{z}_1$ is an $\boldsymbol{m}, d$ point for F. We want to solve the equation:

$$DF^d_{\boldsymbol{z}_1}(v) = \lambda v \tag{31.1}$$

with $v \in T(T^*\mathbb{T}^n)_{\boldsymbol{z}_1}$. We follow MacKay & Meiss (1983): If $\overline{\boldsymbol{q}}$ corresponds to the orbit of $\boldsymbol{z}_1$ under the the successive F_k's, it must satisfy:

$$\frac{\partial W(\overline{\boldsymbol{q}})}{\partial \boldsymbol{q}_k} = \partial_2 S_{k-1}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k) + \partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) = 0.$$

Therefore, a "tangent orbit" $\delta\overline{\boldsymbol{q}}$ must satisfy:

$$S^{k-1}_{21}\delta\boldsymbol{q}_{k-1} + (S^k_{11} + S^{k-1}_{22})\delta\boldsymbol{q}_k + S^k_{12}\delta\boldsymbol{q}_{k+1} = 0 \tag{31.2}$$

where we have abbreviated:

$$S^k_{ij} = \partial_{ij} S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}).$$

Remark 31.3 This rather physical argument can be given a more mathematical footing. Consider the following:

$$\begin{aligned} T^*\mathbb{R}^n &\cong \left\{((\boldsymbol{q}_1, \boldsymbol{p}_1), \ldots, (\boldsymbol{q}_{dN}, \boldsymbol{p}_{dN}) \in (T^*\mathbb{R}^n)^{dN} \mid F_k(\boldsymbol{q}_k, \boldsymbol{p}_k) = (\boldsymbol{q}_{k+1}, \boldsymbol{p}_{k+1})\right\} \\ &\cong \{\overline{\boldsymbol{q}} \in (\mathbb{R}^n)^{dN+1} \mid \nabla W(\overline{\boldsymbol{q}})_k = 0, k = 1, \ldots, dN-1\} \end{aligned}$$

The first homeomorphism is between points in the space and their orbit segments of a given length, the second is given by the correspondence between orbit segments and critical points of the action. If one expresses a parametrization of an element of $T(T^*\mathbb{R}^n)$ with the first representation, one gets the orbit of a tangent vector under the differentials of the F_k's. If one uses the second identification , one gets (31.2) .

When $\overline{q}$ corresponds to a periodic point $(\boldsymbol{q}_1, \boldsymbol{p}_1)$, Equation (31.1) translates, in terms of the $\delta\overline{\boldsymbol{q}}$, to:

$$\delta \boldsymbol{q}_{dN+1} = \lambda \delta \boldsymbol{q}_1 \tag{31.3}$$

Equations (31.2) , (31.3) can be put in matrix form as $M(\lambda)\delta\overline{\boldsymbol{q}} = 0$ where $M(\lambda)$ is the following $dNn \times dNn$ matrix:

$$M(\lambda) = \begin{pmatrix} S_{22}^{dN} + S_{11}^1 & S_{12}^1 & 0 & \dots & 0 & \frac{1}{\lambda} S_{21}^{dN} \\ S_{21}^1 & S_{22}^1 + S_{11}^2 & S_{12}^2 & \ddots & & 0 \\ 0 & S_{12}^2 & & & & \vdots \\ \vdots & \ddots & & & & 0 \\ 0 & \dots & 0 & & & S_{12}^{dN-1} \\ \lambda S_{12}^{dN} & 0 & \dots & 0 & S_{21}^{dN-1} & S_{22}^{dN-1} + S_{11}^{dN} \end{pmatrix}$$

(each entries represents an $n \times n$ matrix.) Hence the eigenvalues of $DF_{\boldsymbol{z}_1}^d$ are in one to one correspondence with the values λ for which $\det M(\lambda) = 0$. More precisely, to each eigenvector of $DF_{\boldsymbol{z}_1}^d$ corresponds one and only one vector $\delta\overline{\boldsymbol{q}}$ solution of $M(\lambda)\delta\overline{\boldsymbol{q}} = 0$. Setting $\lambda = 1$, we get $M(1) = \nabla^2 W$, which finishes the proof. □

Remark 31.4 The above relationship between eigenvalues of DF^d and of $\nabla^2 W$ can be given a symplectic interpretation: the Lagrangian manifolds graph(dW) and graph(F) are related by symplectic reduction. Lemma 31.2can then be restated in terms of the invariance of a certain Maslov index under reduction Viterbo (1987).

Lemma 31.2 proves in particular that the condition "all $\boldsymbol{m}, d$ orbits are nondegenerate" is equivalent to "W is a Morse function". The following proposition shows that both properties are true for generic symplectic twist maps .

Proposition 31.5 *For generic symplectic twist maps , all periodic orbits are nondegenerate and hence all the functions W are Morse*

Proof. We remind the reader that a property is *generic* on a topological space if it satisfied on a *residual* set of that space, *i.e.* a countable intersection of open and dense sets. Robinson Robinson (???), in his theorem 1Bi, proves that the set of C^k symplectic maps with nondegenerate periodic points is residual in the space of all C^k symplectic maps. He proceeds by induction on the period d of the points[(6)]. We want to adapt his proof to the space STM of C^1 of symplectic twist maps . First note that, since the twist condition is open, STM is an open set in the space of C^1 exact symplectic maps. The only thing that we have to check, therefore, is that the perturbations that Robinson uses to kill degeneracy transform exact symplectic maps into exact symplectic maps. But this is not hard to check: each of these perturbations is given by composing the original map f with the time one map of the hamiltonian flow associated to a bump function in a small neibourghood of a given periodic point. Hence the perturbed map is the composition of the original exact symplectic map with the time 1 map of a Hamiltonian, also exact symplectic by Theorem 47.7. The composition of two exact symplectic maps being exact symplectic, we are done. □

[6] C.Robinson actually deals with higher order resonnances as well, i.e, roots of unity in the spectrum of $Df_{\boldsymbol{z}}^d$.

32. The Convex Case

The standing assumption in this section is that $F = F_N \circ \ldots \circ F_1$ where F_k is a symplectic twist map with generating function S_k satisfying the convexity condition:

$$\langle \partial_{12} S_k(\boldsymbol{q}, \boldsymbol{Q}).\boldsymbol{v}, \boldsymbol{v} \rangle \leq -a \|\boldsymbol{v}\|^2, \quad \forall \boldsymbol{q}, \boldsymbol{Q}, \boldsymbol{v} \in \mathbb{R}^n, k \in \{1, \ldots, N\}. \tag{29.1}$$

The central part of the proof of the convex case, due to Kook & Meiss (1989)consists in proving that the function W is proper, and hence has a minimum. This is something we have already done in the case $n = 1$ (see), and the proof in higher dimensions is identical. (??? Change this sentence if I put the min part of AM in a MIN chapter)

Lemma 32.1 *Let S be the generating function of a symplectic twist map satisfying the convexity condition. Then there is an α and positive β and γ such that:*

$$S(\boldsymbol{q}, \boldsymbol{Q}) \geq \alpha - \beta \|\boldsymbol{q} - \boldsymbol{Q}\| + \gamma \|\boldsymbol{q} - \boldsymbol{Q}\|^2 . \tag{32.1}$$

Corollary 32.2 *Let F satisfy the convexity condition (29.2) . Then there is a minimum for the corresponding action function W (and hence an $\boldsymbol{m}, d$–point for F.)*

We have thus found at least one $\boldsymbol{m}, d$–orbit corresponding to a minimum of W. The reader should be aware that, unlike the 1 degree of freedom case, this does not imply that the orbit is a global minimizer (see Hermann (1990) and Arnaud (1989)).

We now turn to the multiplicity of orbits.

This proof can be rewritten using Conley theory only. I should do that if I'm going to get rid of the section on Morse theory in Appendix 2 or TOPO... Outline: Use 51.1(about the retraction): The isolating block W^K with empty exit set, so $H^*(W^K, (W^K)^-) \equiv H^*(W^K)$. Also there is the requisite retraction....

Remember that $\mathbf{X}$ is a bundle over $\mathbb{T}^n$. Let $\Sigma \cong \mathbb{T}^n$ be its zero section. Let $K > \sup_{\overline{\boldsymbol{q}} \in \Sigma} W(\overline{\boldsymbol{q}})$. Trivially, we have:

$$\Sigma \subset W^K \stackrel{\text{def}}{=} \{\overline{\boldsymbol{q}} \in \mathbf{X} \mid W \leq K\}$$

(since W is proper, for almost every K, W^K is a compact manifold with boundary, by Sard's Theorem.) From this we get the commutative diagram in homology:

$$\begin{array}{ccc} H_*(\Sigma) & \xrightarrow{\quad k_* \quad} & H_*(\mathbf{X}) \\ i_* \searrow & & \nearrow j_* \\ & H_*(W^K) & \end{array} \tag{32.2}$$

where i, j, k are all inclusion maps. But $k_* = Id$ since Σ and $\mathbf{X}$ have the same homotopy type. Hence i_* must be injective.

If all the $\boldsymbol{m}, d$–points are nondegenerate, W is a Morse function (a generic situation by Proposition 32.0) and according to Morse Theory (Milnor (1969) , Section 3) W^K has the homotopy type of a finite CW complex, with one cell of dimension k for each critical point of index k in W^K. In particular, we have the following Morse inequalities:

$$\#\{\text{critical points of index } k\} \geq b_k$$

where b_k is the kth Betti number of W^K, $b_k \geq \binom{n}{k}$ in our case since $H_*(\mathbb{T}^n) \hookrightarrow H_*(W^K)$. Hence there are at least 2^n critical points in this nondegenerate case.

If W is not a Morse function, rewrite the diagram (32.2) , but in cohomology, reversing the arrows and raising the stars. Since $k^* = Id$, j^* must be injective this time. We know that the cup length $cl(X) = cl(\mathbb{T}^n) = n+1$. By definition, this means that there are n cohomology classes $\alpha_1, \ldots, \alpha_n$ in $H^1(\mathbf{X})$ such that $\alpha_1 \cup \ldots \cup \alpha_n \neq 0$. Since j^* is injective, $j^*\alpha_1 \cup \ldots \cup j^*\alpha_n \neq 0$ and thus $cl(W^K) \geq n+1$. W^K being compact, and invariant under the gradient flow, Lusternik-Schnirelman theory implies that W has at least $n+1$ critical points in W^K (The proof of Theorem 1 in CH.2 Section19 of Dubrovin & al. (1987) , which is for compact manifolds without boundaries can easily be adapted to this case.) □

33. Asymptotically Linear Systems

In this section we swap the convexity condition (29.1) for asymptotic linearity of the map (29.2) . In this case, the periodic action function W does not necessarily have any minimum. The topological tool we use here is Proposition 52.8.

We remind our reader of our assymption (29.2) : $F = F_N \circ \ldots \circ F_1$ is a product of lifts of symplectic twist maps of $T^*\mathbb{T}^n$. The generating function S_k of F_k satisfies:

$$S_k(\boldsymbol{q}, \boldsymbol{Q}) = \frac{1}{2}\langle A_k(\boldsymbol{Q} - \boldsymbol{q}), (\boldsymbol{Q} - \boldsymbol{q})\rangle + R_k(\boldsymbol{q}, \boldsymbol{Q})$$

with:

$$(29.2) \qquad A_k = A_k^t,\ \det\ A_k \neq 0, \quad \det \sum_1^N A_k^{-1} \neq 0, \qquad \lim_{\|\boldsymbol{Q}-\boldsymbol{q}\| \to \infty} \frac{\nabla R_k(\boldsymbol{q}, \boldsymbol{Q})}{\|\boldsymbol{Q} - \boldsymbol{q}\|} = 0$$

We view R as a *global* perturbation term. As before we let $L_k(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + A_k^{-1}\boldsymbol{p})$ and $L = L_N \circ \ldots \circ L_1$. Then $L(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + A\boldsymbol{p})$ with $A = \sum_1^N A_k^{-1}$. L and all the L_k's are completely integrable symplectic twist maps .

As before, we are looking for critical points of:

$$W(\overline{\boldsymbol{q}}) = \sum_{k=1}^{dN} S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) = \sum_{k=1}^{dN} \frac{1}{2} \left\langle A_k(\boldsymbol{q}_{k+1} - \boldsymbol{q}_k), (\boldsymbol{q}_{k+1} - \boldsymbol{q}_k)\right\rangle + \sum_{k=1}^{dN} R_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}).$$

where $\overline{\boldsymbol{q}} \in \overline{\mathbf{X}}$ i.e., $\boldsymbol{q}_{dN+1} = \boldsymbol{q}_1$. The first sum in the right hand side is quadratic, call it $\mathcal{Q}'$. It is the action function for the symplectic twist map L defined above. We change coordinates $\Psi : (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{dN-1}) \mapsto (\boldsymbol{q}, \boldsymbol{v})$ as in Section 30:

$$\boldsymbol{q} = \frac{1}{dN} \sum_1^{dN} \boldsymbol{q}_k$$

$$\boldsymbol{v}_k = \boldsymbol{q}_{k+1} - \boldsymbol{q}_k - \boldsymbol{m}/dN, \quad k \in \{1, \ldots, dN-1\}.$$

In these coordinates, W is of the form:

$$W(\boldsymbol{q}, \boldsymbol{v}) = \mathcal{Q}(\boldsymbol{v}) + R(\boldsymbol{q}, \boldsymbol{v})$$

where $\mathcal{Q}$ is the homogeneous quadratic function:

$$\mathcal{Q}(\boldsymbol{v}) = -\frac{1}{2}\left\langle A_{dN}(\sum_{1}^{dN-1} \boldsymbol{v}_k), \sum_{1}^{dN-1} \boldsymbol{v}_k)\right\rangle + \frac{1}{2}\sum_{k=1}^{dN-1} \langle A_k \boldsymbol{v}_k, \boldsymbol{v}_k \rangle \,,$$

and $R = \sum_1^{dN} R_k \circ \Psi^{-1}$. Postponing the proof that $\mathcal{Q}(v)$ is nondegenerate, we conclude the proof of the theorem.

The maps τ_n and σ introduced in Section 30 all map fibers to fibers diffeomorphically and linearly in the trivial bundle $\overline{\mathbf{X}} \to \mathbb{R}^n$ with projection $(\boldsymbol{q}, \boldsymbol{v}) \mapsto \boldsymbol{q}$. Hence $\mathcal{Q}(\boldsymbol{q}, \boldsymbol{v}) = \mathcal{Q}(\boldsymbol{v})$ which is quadratic nondegenerate in the fibers induces in the quotient $\mathbf{X}$ of $\overline{\mathbf{X}}$ a function $\mathcal{Q}$ which is also quadratic nondegenerate in the fibers of the bundle $\mathbf{X} \to \mathbb{T}^n$. Finally, it is easy to see that the asymptotic condition on R_k given in (29.2) implies that:

$$\frac{1}{\|\boldsymbol{v}\|}\frac{\partial}{\partial \boldsymbol{v}}(W - \mathcal{Q}) = \frac{1}{\|\boldsymbol{v}\|}\frac{\partial R}{\partial \boldsymbol{v}} \to 0 \quad \text{as} \quad \|\boldsymbol{v}\| \to \infty$$

in $\overline{\mathbf{X}}$ and hence also in its quotient $\mathbf{X}$. We apply Proposition gpqi to conclude the proof of Theorem 29.1.

We now turn to the proof that, given the asumption (29.2) , $\mathcal{Q}(\boldsymbol{v})$ is nondegenerate. The reader could work the linear algebra out directly. We prefer to give a dynamical argument which might enlight us a bit on the linear asymptotic condition. Critical points of $\boldsymbol{v} \mapsto \mathcal{Q}(\boldsymbol{v})$ form the kernel of $\mathcal{Q}$. On the other hand, critical points of $(\boldsymbol{q}, \boldsymbol{v}) \mapsto \mathcal{Q}(\boldsymbol{q}, \boldsymbol{v}) = \mathcal{Q}(\boldsymbol{v})$ are in one to one correspondence with the $\boldsymbol{m}, d$ orbits of the linear map L. Since L is a linear completely integrable symplectic twist map , these orbits form an n dimensional plane parallele to the 0 section of $T^*\mathbb{T}^n$. Since the generating function of L is quadratic and the above change of coordinate Ψ is affine, this plane corresponds 1-1 to an n-plane of critical points of $\mathcal{Q}(\boldsymbol{q}, \boldsymbol{v})$ in $\overline{\mathbf{X}}$. But the n-plane $\{\boldsymbol{v} = 0\}$ is made of critical points of $\mathcal{Q}(\boldsymbol{q}, \boldsymbol{v})$. Therefore, there cannot be any other critical points for $\mathcal{Q}(\boldsymbol{q}, \boldsymbol{v})$, and hence $\mathcal{Q}(\boldsymbol{v})$ has trivial kernel. □

34. Ghost Tori

Let F be as in Theorem (29.2) , and W be the corresponding action function for $\boldsymbol{m}, d$ orbits on $\mathbf{X}$. In the proof of Theorem 29.1(with the asymptotically quadratic condition), we showed that the set of bounded solutions $G = G_1$ of the gradient flow of W continues, in the sense of Conley, the one for the completely integrable map with action function W_0, and that:

$$H^*(G_0) = H^*(\mathbb{T}^n) \hookrightarrow H^*(G)$$

where G_0 is the torus made of critical points of W_0.

Definition 34.1 Let W the action function for a compostion of symplectic twist map $F = F_N \circ \ldots \circ F_1$ on the space $\mathbf{X}$ of $\boldsymbol{m}, d$ sequences. A set G in $\mathbf{X}$ is called a *ghost torus* if it is compact, invariant by the gradient flow of W and if:

$$H^*(\mathbb{T}^n) \hookrightarrow H^*(G).$$

Comments 34.2

(a) If F has an invariant torus made of $\boldsymbol{m}, d$ periodic orbits, the orbit of each point on it corresponds to a critical point in $\mathbf{X}$. Hence the map invariant torus is diffeomorphic to a torus of critical points in $\mathbf{X}$, which is trivially invariant under the gradient flow of W. This torus is hence a ghost torus, we will call it a *completely critical ghost torus* (see Exercise 34.3.)

(b) The spooky connotation in the terminology "ghost tori" can be justified in the following way. One of the essential avenues for the study of symplectic twist maps is the standard family, which fits quite well in the setting of Theorem ???. The paradigm expressed by the standard family is that of a deformation of an integrable map F_0. We have seen that to such a map corresponds a foliation of $T^*\mathbb{T}^n$ by invariant tori, one for each rotation vector. In particular there is exactly one $\boldsymbol{m}, d$ periodic invariant torus for F_0, corresponding to a completely critical ghost torus in the space $\mathbf{X}$ for each $\boldsymbol{m}, d$. One of the fundamental questions in the theory is to understand what happens to these invariant tori as one deforms F_0. ???By now, this should have been stated a hundred times already??? What Theorem ??? shows is that a "ghost" of the invariant torus for F_0 remains, as the parameter s varies, namely G_s, but in the space $\mathbf{X}$. This ghost torus is invariant by the gradient flow of W_s, *but does not necessarily corresponds to an F_s–invariant torus anymore*. Indeed, generically, the only dynamically "visible" part of G_s is formed by the (at least 2^n, but finite number of) critical points that it contains, which correspond to the $\boldsymbol{m}, d$ periodic orbits. G_s is in fact a collection of critical points for W_s and their connecting orbits for the gradient flow : intersections of stable and unstable manifolds for the critical points (this is true of any compact invariant set for a gradient flow.) Here is a table that might be helpful in understanding the analogy we are trying to draw:

Silly Table

Real World	$T^*\mathbb{T}^n, \quad F$
Yonder World	$\mathbf{X} \quad W$
Live Being	Invariant Torus for F
Ghost	Ghost Torus G for $\frac{d}{dt}\overline{\boldsymbol{q}} = \nabla W(\overline{\boldsymbol{q}})$
Soul	$H^*(\mathbb{T}^n) \hookrightarrow H^*(G)$
Time	Parameter in the Standard Map
Transcending	Map $\mathcal{T}$ from $T^*\mathbb{T}^n$ to $\mathbf{X}$:
	$\mathcal{T}(\boldsymbol{q}_1, \boldsymbol{p}_1) = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{dN})$, where $(\boldsymbol{q}_{k+1}, \boldsymbol{p}_{k+1}) = F_k(\boldsymbol{q}_k, \boldsymbol{p}_k)$.
Appearing	Map $\mathcal{A}$ from $\mathbf{X}$ to $T^*\mathbb{T}^n$:
	$\mathcal{A}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{dN}) = (\boldsymbol{q}_1, \boldsymbol{p}_1(\boldsymbol{q}_1, \boldsymbol{q}_2))$.

(c) Instead of thinking of G_s as a subset of $\mathbf{X}$, one can remember that the set G_s is the projection of the τ and σ^N invariant set gG_s in $\overline{\mathbf{X}} \subset (\mathbb{R}^n)^{\mathbb{Z}}$.

(d) If F is as in Theorem 29.1(convex case), one can reword the proof of that theorem in order to deduce the existence of a ghost torus: we have shown in ??? that a map satisfying the convexity condition ??? could be deformed to a completely integrable one, through a path of symplectic twist maps satisfying this condition. Let F_s be such a path and W_s the corresponding action function. Since we have seen in the proof of Theorem ??? that they were no critical points outside of a set W_s^K for K big enough (we can make K uniform in $s \in [0, 1]$), the set G_s of bounded solutions for the gradient flow of W must be included in W_s^K and thus (see ???) the sets G_s are related by continuation. G_0 is normally hyperbolic, as in the proof of Theorem ??? and thus we can conclude this alternate proof of Theorem ??? (convex case) as in ??? (formno...), and in particular $G = G_1$ is a ghost torus for $F = F_1$.

(e) Ghost tori are quite reminiscent of the set of connecting orbits that supports Floer's homology complex , as it is applied to Hamiltonian systems on the cotangent bundle of $\mathbb{T}^n$ (the space that Cieleback (1992)calls X in .) It is quite probable that, at least at the (co)homology level, when the map F is Hamiltonian and satisfies the hypothesis of Theorem ???, these sets are identical.

(f) We put the title "Rational ghost tori" to this section, because they live in spaces of sequences with rotation vector m/d. We will discuss later the occurence of irrational ghost tori ???, and their connection with the KAM and Aubry–Mather theory.

Exercise 34.3 Show that the Transcendence of an F–invariant torus is a completely critical ghost torus. Show that one is a Live Being if and only if one is the Appearence of one's own Transcendence. In general, reread the previous paragraphs and give them more rigorous sense with the help of the maps $\mathcal{A}$ and $\mathcal{T}$.

Theorem thesis is 29.1

Condition STMPtquad is (29.2)

Lemma STMPlemsecvar is 31.2, Proposition 31.5is STMPpropgeneric Condition STMPconv is (29.1) , Lemma STMPlemquadconv is 32.1