Chapter 4 or STM

SYMPLECTIC TWIST MAPS

9/25/99

This is the version revised on January 12 1998. It minimizes the use of symplectic theory or of homotopy (in the torus case). The general case could be moved to another chapter. Birkhoff-Lewis: point to the idea of proof: intersection of Lag tori. Get the hyperbolic metric right. Find page in Gallot on diffeo $TM \cong M$. Find the whereabouts of Eduardo's picture (which map, which orbit). State Birkhoff's normal form for invariant diophantine tori (see Yoccoz, page 754-07, Herman IMA?)

In this chapter, we generalize the definition of twist maps of the annulus to that of symplectic twist maps in higher dimensions. In many cases, around elliptic fixed points, area preserving planar maps yield twist maps of the annulus $\mathbb{S}^1 \times \mathbb{R}$. Likewise, symplectic maps in $\mathbb{R}2n$ around their elliptic fixed points lead to symplectic twist maps of $\mathbb{T}^n \times \mathbb{R}^n$, the cotangent bundle of the n dimensional torus. This is one among many other reasons which make $\mathbb{T}^n \times \mathbb{R}^n$ one of the most natural spaces to study. Another reason is that, although these notions are at least implicitly present, almost no knowledge of manifolds, fiber bundles and differential forms is needed for the study of symplectic maps on this space. Hence we devote the first sections of this chapter to defining symplectic twist map of $\mathbb{T}^n \times \mathbb{R}^n$ and exploring their relationship with their generating functions.

Nonetheless, cotangent bundles of many other manifolds do occur in mechanics (*eg.* the configuration space of the solid rigid body is SO(3)) and there too it is possible to define and make use of symplectic twist maps. For this part of the chapter, the reader should be familiar with the notion of cotangent bundle, differential forms as are given in Section 46 of Appendix 1 or SG.

23. Symplectic Twist Maps of $\mathbb{T}^n \times \mathbb{R}^n$

A. Definition

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n–dimensional torus. An analog to the annulus in higher dimensions which is most natural in mechanics is the space $\mathbb{T}^n \times \mathbb{R}^n$, which can be seen as the cartesian product of n annuli. We give $\mathbb{T}^n \times \mathbb{R}^n$ the coordinate $(\boldsymbol{q}, \boldsymbol{p}) = (q_1, \dots, q_n, p_1, \dots, p_n)$. In mechanics, $q_1, \dots, q_n$ would be n angular configuration variables of the system, whereas $p_1, \dots, p_n$ would be their conjugate momentum, and $\mathbb{T}^n \times \mathbb{R}^n$ is the cotangent bundle $T^*\mathbb{T}^n$ of the torus $\mathbb{T}^n$.

The following is a generalization of the definition of twist maps of the cylinder:

Definition 23.1 Let F be a diffeomorphism of $\mathbb{R}2n$ and write $(\boldsymbol{Q}(\boldsymbol{q},\boldsymbol{p}),\boldsymbol{P}(\boldsymbol{q},\boldsymbol{p})) = F(\boldsymbol{q},\boldsymbol{p})$. Let F satisfies:

1) $F(\boldsymbol{q}+\boldsymbol{m},\boldsymbol{p}) = F(\boldsymbol{q},\boldsymbol{p}) + (\boldsymbol{m},0)$
2) *Twist Condition:* the map $\psi_F : (\boldsymbol{q},\boldsymbol{p}) \mapsto (\boldsymbol{q},\boldsymbol{Q}(\boldsymbol{q},\boldsymbol{p}))$ is a diffeomorphism of $\mathbb{R}2n$.
3) *Exact Symplectic:* In the coordinates $(\boldsymbol{q},\boldsymbol{Q})$,

$$\boldsymbol{P}d\boldsymbol{Q} - \boldsymbol{p}d\boldsymbol{q} = dS(\boldsymbol{q},\boldsymbol{Q}) \tag{23.1}$$

where S is a real valued function on $\mathbb{R}2n$ satisfying:

$$S(\boldsymbol{q}+\boldsymbol{m},\boldsymbol{Q}+\boldsymbol{m}) = S(\boldsymbol{q},\boldsymbol{Q}), \quad \forall \boldsymbol{m} \in \mathbb{Z}^n. \tag{23.2}$$

Then the map f that F induces on $\mathbb{T}^n \times \mathbb{R}^n$ is called a *Symplectic Twist Map*.

As for maps of the annulus, $S(\boldsymbol{q},\boldsymbol{Q})$ is called a *generating function* of the map F: Equation (23.1) is equivalent to

$$\begin{aligned} \boldsymbol{p} &= -\,\partial_1 S(\boldsymbol{q},\boldsymbol{Q})) \\ \boldsymbol{P} &= \partial_2 S(\boldsymbol{q},\boldsymbol{Q}), \end{aligned} \tag{23.3}$$

and thus F is implicitely given by S since

$$\begin{aligned} F(\boldsymbol{q},\boldsymbol{p}) &= (\boldsymbol{Q}\circ\psi_F(\boldsymbol{q},\boldsymbol{p}), \partial_2 S\circ\psi_F(\boldsymbol{q},\boldsymbol{p})) \quad \text{with} \\ \psi_F^{-1}(\boldsymbol{q},\boldsymbol{Q}) &= (\boldsymbol{q}, -\partial_1 S(\boldsymbol{q},\boldsymbol{Q})) \end{aligned} \tag{23.4}$$

Note that the prescription of F through its generating function S is often more theoretical than computational: it involves the inversion of the diffeomorphism ψ_F^{-1}.

B. Comments on the Definition

(1) The periodicity condition $F(\boldsymbol{q}+\boldsymbol{m},\boldsymbol{p}) = F(\boldsymbol{q},\boldsymbol{p})+(\boldsymbol{m},0)$ implies that F induces a map f on $\mathbb{T}^n\times\mathbb{R}^n$. It also implies that (in fact is equivalent to) f is homotopic to Id (see the Exercise 23.1).

(2) The twist condition (2) of definition 23.0 implies the local twist condition often used in the litterature:

$$\text{Condition}(2') \qquad \det\ \partial\boldsymbol{Q}/\partial\boldsymbol{p} \neq 0,$$

We will explore in Section 26extra assumptions under which the local twist implies the global twist of Condition (2).

(3) In terms of differential forms, $\boldsymbol{P}d\boldsymbol{Q}-\boldsymbol{p}d\boldsymbol{q} = F^*\boldsymbol{p}d\boldsymbol{q}-\boldsymbol{p}d\boldsymbol{q}$. The periodicity of S given by $S(\boldsymbol{q}+\boldsymbol{m},\boldsymbol{Q}+\boldsymbol{m}) = S(\boldsymbol{q},\boldsymbol{Q})$ in the $(\boldsymbol{q},\boldsymbol{Q})$ coordinates becomes $S(\boldsymbol{q}+m,\boldsymbol{p}) = S(\boldsymbol{q},\boldsymbol{p})$ in the $(\boldsymbol{q},\boldsymbol{p})$ coordinates ($i.e.$ applying Ψ_F^{-1}). In particular S induces a function s on $\mathbb{T}^n\times\mathbb{R}^n$ such that $f^*\boldsymbol{p}d\boldsymbol{q}-\boldsymbol{p}d\boldsymbol{q} = ds$ ($\boldsymbol{q}$ is seen as coordinate on $\mathbb{T}^n$ here). This last equality expresses the fact that f is *exact symplectic*. As is made more precise in Chapter SG, if f is exact symplectic it is also *symplectic*:

$$f^*\ pd\boldsymbol{q} - \boldsymbol{p}d\boldsymbol{q} = ds \Rightarrow d(f^*\boldsymbol{p}d\boldsymbol{q} - \boldsymbol{p}d\boldsymbol{q}) = 0 \Rightarrow f^*d\boldsymbol{p}\wedge d\boldsymbol{q} = d\boldsymbol{p}\wedge d\boldsymbol{q}.$$

Any symplectic map of $\mathbb{R}2n$ is exact symplectic, but it is not true of maps of $\mathbb{T}^n\times\mathbb{R}^n$: the map $f(\boldsymbol{q},\boldsymbol{p}) \mapsto (\boldsymbol{q},\boldsymbol{p}+\boldsymbol{m}), \boldsymbol{m}\neq 0$ is symplectic but not exact symplectic. As for maps of the annulus, exact symplecticity can be interpreted as a zero flux condition, but the flux is now an n dimensional quantity.

Exercise 23.2 Each homeomorphism of the torus $\mathbb{T}^n$ is homotopic to a unique torus map induced by a linear map A of $Gl(n, \mathbb{Z})$ (the group of invertible integer $n \times n$ matrices). Likewise, each homotopy classes of homeomorphisms of $\mathbb{T}^n \times \mathbb{R}^n$ has exactly one representant of the form $A \times Id$ where $A \in Gl(n, \mathbb{Z})$. Show that any lift F of a map homotopic to $A \times Id$ satisfies:

$$F(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P}) \Rightarrow F(\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{p}) = (\boldsymbol{Q} + A\boldsymbol{m}, \boldsymbol{P})$$

.

Exercise 23.3 Show that if $F(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P})$ is the lift of a symplectic twist map with generating function $S(\boldsymbol{q}, \boldsymbol{Q})$, then $F^{-1}(\boldsymbol{Q}, \boldsymbol{P}) = (\boldsymbol{q}, \boldsymbol{p})$ is also the lift of a symplectic twist map with generating function $-S(\boldsymbol{Q}, \boldsymbol{q})$.

Exercise 23.4 Show that if F and F' are two lifts of the same symplectic twist map F, their corresponding generating functions S and S' satisfy:

$$S(\boldsymbol{q}, \boldsymbol{Q}) = S'(\boldsymbol{q}, \boldsymbol{Q} + \boldsymbol{m}),$$

where $\boldsymbol{m} \in \mathbb{Z}^n$ is such that $F' = T_{\boldsymbol{m}} \circ F$.

C. The Variational Setting

As in the case of monotone twist maps of the annulus, the generating function of a symplectic twist map induces a variational approach to finding orbits of the map.

Proposition 24.1 (Critical Action Principle) *Let $f_1, \dots, f_N$ be symplectic twist maps of $T^*\mathbb{T}^n$, and let F_k be a lift of F_k, with generating function S_k. There is a one to one correspondence between orbits segments $\{(\boldsymbol{q}_{k+1}, \boldsymbol{p}_{k+1}) = F_k(\boldsymbol{q}_k, \boldsymbol{p}_k)\}$ under the successive F_k's and the sequences $\{\boldsymbol{q}_k\}_{k \in \mathbb{Z}}$ in $(\mathbb{R}^n)^{\mathbb{Z}}$ satisfying:*

$$\partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) + \partial_2 S_{k-1}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k) = 0 \tag{24.1}$$

The correspondence is given by: $\boldsymbol{p}_k = -\partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$.

Proof. It is identical to the case $n = 1$, Corollary 5.2. □

As in the case $n = 1$, Equation (24.1) can be interpreted as:

$$\nabla W(\overline{\boldsymbol{q}}) = 0 \qquad \text{with}$$
$$W(\overline{\boldsymbol{q}}) = \sum_0^{N-1} S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}).$$

25. Examples

Example 25.1 The Generalized Standard Map

The *generalized standard map* or *standard family* is the family of symplectic twist map whose lift is generated by the following functions:

$$S_\lambda(\boldsymbol{q}, \boldsymbol{Q}) = \frac{1}{2}\left\|\boldsymbol{Q} - \boldsymbol{q}\right\|^2 + V_\lambda(\boldsymbol{q}).$$

where V_λ is a family of C^2 functions that are $\mathbb{Z}^n$–periodic, λ a parameter on some euclidian space and $V_0 \equiv 0$. It is trivial to see that S satisfies the periodicity condition $S_\lambda(\boldsymbol{q}+\boldsymbol{m}, \boldsymbol{Q}+\boldsymbol{m}) = S_\lambda(\boldsymbol{q}, \boldsymbol{Q})$. To find the corresponding map, we compute:

$$\begin{aligned}\boldsymbol{p} &= -\partial_1 S_\lambda(\boldsymbol{q}, \boldsymbol{Q}) = \boldsymbol{Q} - \boldsymbol{q} - \nabla V_\lambda(\boldsymbol{q})\\ \boldsymbol{P} &= \partial_2 S_\lambda(\boldsymbol{q}, \boldsymbol{Q}) = \boldsymbol{Q} - \boldsymbol{q}\end{aligned}$$

from which we immediately get:

$$\begin{aligned}\boldsymbol{Q} &= \boldsymbol{q} + \boldsymbol{p} + \nabla V_\lambda(\boldsymbol{q})\\ \boldsymbol{P} &= \boldsymbol{p} + \nabla V_\lambda(\boldsymbol{q})\end{aligned}$$

In other words, the standard map is given by:

$$F_\lambda(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + \boldsymbol{p} + \nabla V_\lambda(\boldsymbol{q}), \boldsymbol{p} + \nabla V_\lambda(\boldsymbol{q})). \tag{25.1}$$

In the case $n = 2$, the following is the most widely studied potential. It is due to Froeschlé (1972)(see also Kook & Meiss (1989), Froeschlé & Laskar (199???)):

$$V_\lambda(q_1, q_2) = \frac{1}{(2\pi)^2}\{K_1 \cos(2\pi q_1) + K_2 \cos(2\pi q_2) + h\cos(2\pi(q_1 + q_2))\}.$$

In this case $\lambda = (K_1, K_2, h) \in \mathbb{R}^3$, and the standard family attached to this potential is a three parameter family of symplectic maps of $\mathbb{T}^2 \times \mathbb{R}^2$. The picture on the bookcover represents the stable and unstable manifolds of a periodic orbit ??? for this map, with parameter???.

When $\lambda = 0$, the map F_λ of (25.1) becomes:

$$F_0(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + \boldsymbol{p}, \boldsymbol{p}).$$

This is an instance of a *completely integrable* symplectic twist map: such maps preserve a foliation of $\mathbb{T}^n \times \mathbb{R}^n$ by tori homotopic to $\mathbb{T}^n \times \{0\}$. On the covering space of each of these tori, the lift of the map is conjugated to a rigid translation. The term "completely integrable" comes from the corresponding notion in Hamiltonian systems (see Example 25.3.)

The reason why the standard map has attracted so much research is that it is a *computable* example in which one may try to understand questions about persistence of invariant tori as the parameter λ varies away from 0, as well as study the various properties of its periodic orbits.

Examples 25.4 Hamiltonian systems

Historically, symplectic twist map appeared as Poincaré return maps in Hamiltonian systems. We develop this idea in Section 19.

Hamiltonian systems in $T^*\mathbb{T}^n$ have also another way of yielding symplectic twist maps: when restricted to an appropriate domain, the time ϵ map of a Hamiltonian system is often a symplectic twist maps.

As a basic example, the Hamiltonian flow generated by:

$$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2}\langle A\boldsymbol{p}, \boldsymbol{p}\rangle \quad \text{with} \quad A^t = A, \ \det A \neq 0$$

is completely integrable, in that it preserves each torus $\{\boldsymbol{p} = \boldsymbol{p}_0\}$ and its time t map:

$$g^t(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + t(A\boldsymbol{p}), \boldsymbol{p})$$

is a completely integrable symplectic twist map. If A is positive definite, g^t restricted to $\{H = 1\}$ is just the geodesic flow for the flat metric $\frac{1}{2}\langle A^{-1}\boldsymbol{v}, \boldsymbol{v}\rangle$ on $\mathbb{T}^n$. (See 26.)

More generally, if $F(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P})$ is the lift of the time ε of some Hamiltonian function H, then:

$$\begin{aligned} Q &= q(\epsilon) = q(0) + \epsilon . H_p + o(\epsilon^2) \\ P &= p(\epsilon) = p(0) - \epsilon . H_q + o(\epsilon^2), \end{aligned}$$

and F satisfies the local twist condition "$\frac{\partial Q}{\partial p}(z(0))$ is non degenerate" whenever H_{pp} is non degenerate. This remark was made by Moser (1986) in the dimension 2 case. From this local argument we will derive conditions under which the time ε of a Hamiltonian is a symplectic twist map .

We will also see that, even if the time ϵ map of a Hamiltonian system is not twist, its time 1 map can, for large classes of Hamiltonian systems, still be decomposed into the product of twist maps. Chapter 4 explores these issues in detail.

Exercise 25.5 Compute the expression of the lift of a symplectic twist map generated by:

$$S(\boldsymbol{q}, \boldsymbol{Q}) = \frac{1}{2}\langle A(\boldsymbol{Q} - \boldsymbol{q}), (\boldsymbol{Q} - \boldsymbol{q})\rangle + \boldsymbol{c}.(\boldsymbol{Q} - \boldsymbol{q}) + V(\boldsymbol{q}).$$

Where A is a nondegenerate $n \times n$ symmetric matrix. (This is yet a further generalization of the standard map.)

26. More On Generating Functions

In this section, we explore more in detail the relationship between generating functions and symplectic twist maps.

Proposition 26.1 *There is a homeomorphism*[4] *between the set of lifts F of C^1 symplectic twist maps of $T^*\mathbb{T}^n$ and the set of C^2 real valued functions S on $\mathbb{R}2n$ satisfying the following:*

(a) $S(\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{Q} + \boldsymbol{m}) = S(\boldsymbol{q}, \boldsymbol{Q}), \quad \forall \boldsymbol{m} \in \mathbb{Z}^n$

(b) The maps: $\boldsymbol{q} \to \partial_2 S(\boldsymbol{q}, \boldsymbol{Q}_0)$ and $\boldsymbol{Q} \to \partial_1 S(\boldsymbol{q}_0, \boldsymbol{Q})$ are diffeomorphisms of $\mathbb{R}^n$ for any $\boldsymbol{Q}_0$ and $\boldsymbol{q}_0$ respectively.

(c) $S(0, 0) = 0.$

This correspondence is given by:

$$F(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P}) \Leftrightarrow \begin{cases} \boldsymbol{p} = & -\partial_1 S(\boldsymbol{q}, \boldsymbol{Q})) \\ \boldsymbol{P} = & \partial_2 S(\boldsymbol{q}, \boldsymbol{Q})). \end{cases} \tag{26.1}$$

Proof. Let F be a lift of a symplectic twist map and $S(\boldsymbol{q}, \boldsymbol{Q})$ be its generating function. For such F and S, we have already derived (26.1) from $\boldsymbol{P}d\boldsymbol{Q} - \boldsymbol{p}d\boldsymbol{q} = dS$, and (a) is part of our definition of symplectic twist

[4] In the compact open topologies of the corresponding sets

maps . To show that S satisfies (b), first notice that, by (26.0) , $\boldsymbol{Q} \to -\partial_1 S(\boldsymbol{q}_0, \boldsymbol{Q})$ is just the inverse of the map $\boldsymbol{p} \to \boldsymbol{Q}(\boldsymbol{q}_0, \boldsymbol{p})$, which is a diffeomorphism since $\psi_F : (\boldsymbol{q}, \boldsymbol{p}) \mapsto (\boldsymbol{q}, \boldsymbol{Q})$ is a diffeomorphism by the twist condition. We also have the composition of diffeomorphisms:

$$(\boldsymbol{q}, \boldsymbol{Q}) \stackrel{\psi_F^{-1}}{\to} (\boldsymbol{q}, \boldsymbol{p}) \stackrel{F}{\to} (\boldsymbol{Q}, \boldsymbol{P})$$

which implies that the map $\boldsymbol{q} \to \boldsymbol{P}(\boldsymbol{q}, \boldsymbol{p}_0)$ is a diffeomorphism (that is, F^{-1} satisfies the twist condition), which finishes to prove that S satisfies (b). Since two generating functions of the same F only differ by a constant there is exactly one such $S(0,0) = 0$.

Conversely, given an S satisfying (b), we can define a C^1 exact symplectic map F of $\mathbb{R}2n$ by:

$$(26.2) \qquad \begin{aligned} F(\boldsymbol{q}, \boldsymbol{p}) &= (\boldsymbol{Q} \circ \psi_F(\boldsymbol{q}, \boldsymbol{p}), \partial_2 S \circ \psi_F(\boldsymbol{q}, \boldsymbol{p})) \\ \text{where} \quad \psi_F^{-1}(\boldsymbol{q}, \boldsymbol{Q}) &= (\boldsymbol{q}, -\partial_1 S(\boldsymbol{q}, \boldsymbol{Q})). \end{aligned}$$

It is easy to check that such a pair F, S satisfies (26.1) . Since S satisfies (a), F a lift of a diffeomorphism of $T^*\mathbb{T}^n$: (a) also holds for $\partial_1 S$ and $\partial_2 S$, which implies that $F(\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{p}) = (\boldsymbol{Q} + \boldsymbol{m}, \boldsymbol{P})$ whenever $F(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P})$. Exercise 23.2shows that F must be homotopic to the Identity. Because of (b), F satisfies the twist condition. Hence the map F (uniquely) defined from (26.1) is a symplectic twist map and it is not hard to see that the correspondence we built between the maps F and the functions S is continuous in the C^1 and C^2 compact open topologies respectively. □

In practice, to recognize whether a function S on $\mathbb{R}2n$ is a generating function for some F, it is usefull to have a criterion to decide when S satisfies condition (b) in Proposition 26.0. This is the purpose of the following Propositions:

Proposition 26.2 *Let* $S: \mathbb{R}2n \to \mathbb{R}$ *be a* C^2 *function satisfying:*

$$(26.3) \qquad \begin{aligned} &(i) S(\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{Q} + \boldsymbol{m}) = S(\boldsymbol{q}, \boldsymbol{Q}), \quad \forall \boldsymbol{m} \in \mathbb{Z}^n \\ &(ii) \det \partial_{12} S \neq 0 \\ &(iii) \sup_{(\boldsymbol{q}, \boldsymbol{Q}) \in \mathbb{R}2n} \left\| (\partial_{12} S(\boldsymbol{q}, \boldsymbol{Q}))^{-1} \right\| = K < \infty. \end{aligned}$$

Then S is the generating function for the lift of a symplectic twist map .

Proof. The proof is an immediate consequence of Lemma 26.3 applied to the two maps $\boldsymbol{q} \to \partial_2 S(\boldsymbol{q}, \boldsymbol{Q}_0)$ and $\boldsymbol{Q} \to \partial_1 S(\boldsymbol{q}_0, \boldsymbol{Q})$ (note that $\left\|(\partial_{21} S)^{-1}\right\| = \left\|(\partial_{12} S)^{-1}\right\|$) and of Proposition 26.1.

Lemma 26.3 *Let* $f : \mathbb{R}^N \to \mathbb{R}^N$ *be a local diffeomorphism at each point, such that:*

$$\sup_{x \in \mathbb{R}^N} \left\| (Df_x)^{-1} \right\| = K < \infty.$$

Then f is a global diffeomorphism .

We postpone the proof of this lemma to the end of the section.

□

The following Proposition gives a condition under which the local twist condition can be made global.

Proposition 26.4 *Let $F(\boldsymbol{q},\boldsymbol{p}) = (\boldsymbol{Q},\boldsymbol{P})$ be a symplectic map of* $\mathbb{R}2n$ *with $F(\boldsymbol{q}+\boldsymbol{m},\boldsymbol{p}) = (\boldsymbol{Q}+\boldsymbol{m},\boldsymbol{P})$. Suppose that*

$$\sup_{(\boldsymbol{q},\boldsymbol{p})\in\mathbb{R}2n} \left\|(\partial \boldsymbol{Q}(\boldsymbol{q},\boldsymbol{p})/\partial \boldsymbol{p})^{-1}\right\| < \infty. \tag{26.4}$$

Then F is the lift of a symplectic twist map .

Proof. By Lemma 26.3, for each fixed $\boldsymbol{q}$, the map $\boldsymbol{p} \to \boldsymbol{Q}(\boldsymbol{q},\boldsymbol{p})$ is a global diffeomorphism of $\mathbb{R}^n$. This implies that $\psi_F : (\boldsymbol{q},\boldsymbol{p}) \to (\boldsymbol{q},\boldsymbol{Q})$ is a global diffeomorphism of $\mathbb{R}2n$. □

Proof of Lemma 26.0 We first prove that f is onto. Let $y_0 = f(0)$ and take any $y \in \mathbb{R}^N$. Let $y(t) = (1-t)y_0 + ty$. By the inverse function theorem, f^{-1} is defined and differentiable on an interval $y([0,\epsilon))$. Let a be the supremum of all such ϵ in $[0,1]$. If we prove that f^{-1} is also defined and differentiable at a, then $a = 1$, otherwise, by the inverse function theorem, we get the contradiction that f^{-1} is defined on $[0, a+\alpha)$, for some $\alpha > 0$. For any $t_0, t_1 \in [0,a)$, we have:

$$\begin{aligned}\left\|f^{-1}(y(t_1)) - f^{-1}(y(t_0))\right\| &\le \sup_{t\in[0,a)} \left\|Df^{-1}(y(t))\right\| \|y-y_0\| \, |t_1 - t_0| \\ &\le K \|y - y_0\| \, |t_1 - t_0| \, .\end{aligned}$$

So that, for any sequence $t_k \to a$, the sequence $f^{-1}(y(t_k))$ is Cauchy. This proves the existence of $f^{-1}(y(a))$, which implies that f is onto. Since f is onto and open, it is a covering map from $\mathbb{R}^N$ to $\mathbb{R}^N$. Such a covering has to be one sheeted, since $\mathbb{R}^N$ is connected and simply connected. (See Appendix Covering spaces.) This finishes the proof. □

Finally, we end this section with a useful formula.

Proposition 26.5 *The following formula relates the differential of a symplectic twist map F to the second derivatives of its generating function:*

$$DF_{(\boldsymbol{q},\boldsymbol{p})} = \begin{pmatrix} -\partial_{11}S.(\partial_{12}S)^{-1} & -(\partial_{12}S)^{-1} \\ \partial_{21}S - \partial_{22}S.\partial_{11}S.(\partial_{12}S)^{-1} & -\partial_{22}S.(\partial_{12}S)^{-1} \end{pmatrix}.$$

where all the partial derivatives are taken at the point $(\boldsymbol{q},\boldsymbol{Q}) = \psi_F(\boldsymbol{q},\boldsymbol{p})$.

Proof. We will show that $\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{p}} = -(\partial_{12}S)^{-1}(\boldsymbol{q},\boldsymbol{Q})$, where, as usual, we have set $F(\boldsymbol{q},\boldsymbol{p}) = (\boldsymbol{Q},\boldsymbol{P})$. Differentiating the equality: $\boldsymbol{p} = -\partial_1 S(\boldsymbol{q},\boldsymbol{Q})$ with respect to $\boldsymbol{p}$, viewing $\boldsymbol{Q}$ as a function of $\boldsymbol{q},\boldsymbol{p}$, one gets:

$$Id = -\partial_{12}S(\boldsymbol{q},\boldsymbol{Q}).\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{p}}.$$

The computations for the other terms are similar. □

Exercise 26.6 a) Show that if instead of Condition (1) in the definition of symplectic twist maps we ask F to be homotopic to $A \times Id$, where a lift $\tilde{A}$ of A is in $Gl^+(n,\mathbb{Z})$, then Proposition 26.5 remains true, replacing (a) by:

$$S\left(\boldsymbol{q}+\boldsymbol{m}, \boldsymbol{Q}+\tilde{A}(\boldsymbol{m})\right) = S(\boldsymbol{q},\boldsymbol{Q}).$$

b) Find the map generated by

$$S(\boldsymbol{q}, \boldsymbol{Q}) = \frac{1}{2}(\boldsymbol{q} - \tilde{A}^{-1}\boldsymbol{Q})^2 + V(\boldsymbol{q})$$

Note that this exercise shows, in particular, that there are plenty of examples of exact symplectic maps of $T^*\mathbb{T}^n$ that are not homotopic to Id and hence cannot be Hamiltonian maps.

Exercise 26.7 Let $\mathbb{B}^n$ denote a *compact* ball in $\mathbb{R}^n$. Show that if $f : \mathbb{B}^n \to \mathbb{R}^n$ is a differentiable map satisfying :

$$\inf_{x \in \mathbb{B}^n} \langle df_x \boldsymbol{v}, \boldsymbol{v} \rangle \geq a \langle \boldsymbol{v}, \boldsymbol{v} \rangle, \quad \forall \boldsymbol{v} \in \mathbb{R}^n$$

then f is an embedding (diffeomorphism on its image) of $\mathbb{B}^n$ in $\mathbb{R}^n$.

27. Symplectic Twist Maps on Cotangent Bundles of General Compact Manifolds

If the manifold M is not covered (topologically) by $\mathbb{R}^n$, problems occur when we want to make the definition of symplectic twist maps of T^*M as global as in $T^*\mathbb{T}^n$: there cannot be a global diffeomorphism from a fiber of T^*M to the universal cover $\tilde{M}$. This is why we must restrict ourselves to a neighborhood U of the 0–section in T^*M, feeling free to take $U = T^*M$ whenever possible.

In the following U will denote an open subset of T^*M such that:

$$\pi^{-1}(\boldsymbol{q}) \cap U \simeq interior(\mathbb{B}^n) \tag{27.1}$$

where $\pi : T^*M \to M$ is the canonical projection, and $\mathbb{B}^n \subset \mathbb{R}^n$ denotes the n-ball. Hence U is a ball bundle over M, diffeomorphic to T^*M, but relatively compact in T^*M. In practice , the neighborhood on which we let our maps act will be of the form:

$$U = \{(\boldsymbol{q}, \boldsymbol{p}) \in T^*\tilde{M} \mid H(\boldsymbol{q}, \boldsymbol{p}) < K\}$$

for some function H convex in $\boldsymbol{p}$. When it makes sense, we can let $U = T^*M$ or $U = T^*\tilde{M}$ (e.g., when M is covered by $\mathbb{R}^n$). As in Appendix 1 or SG, we denote by λ the canonical one form on T^*M.

Definition 27.1 A *symplectic twist map* F is a diffeomorphism of an open ball bundle $U \subset T^*M$ (as in (27.1)) onto itself satisfying the following:

(1) F is homotopic to Id.
(2) F is *exact symplectic*: $F^*\lambda - \lambda = \underline{S}$ for some real function valued $\underline{S}$ on U.
(3) (*Twist condition*:) the map $\psi_F : U \to M \times M$ given by $\psi_F(\boldsymbol{z}) = (\pi(\boldsymbol{z}), \pi \circ F(\boldsymbol{z}))$ is an embedding.

The function $S = \underline{S} \circ \psi_F^{-1}$ on $\psi_F(U)$ is called the **generating function** for F.

We leave the reader to check that, in coordinates, this is an obvious generalization of the definition of symplectic twist map of $T^*\mathbb{T}^n$, with the appropriate restrictions of domains. If $\tilde{M} \cong \mathbb{R}^n$, one can take $U = T^*M$ and modify the above definition slightly to make it more global by changing (2) into:

(2') If $\tilde{F} : T^*\tilde{M} \to T^*\tilde{M}$ is a lift of F, the map $\psi_{\tilde{F}} : \tilde{U} \to M \times M$ given by $\psi_{\tilde{F}}(\boldsymbol{z}) = (\pi(\boldsymbol{z}), \pi \circ F(\boldsymbol{z}))$ is a diffeomorphism (of $\mathbb{R}2n$).

It is not hard to adapt the proof of Proposition 26.1 to the more general:

Proposition 27.2 *There is a homeomorphism between the set of pairs (F, U) where F is a C^1 symplectic twist map of $U \subset T^*M$ and the pairs (S, V), where S is in the set of C^2 real valued functions S on an open set V (diffeomorphic to U) of $M \times M$ satisfying the following:*

(i) The map $\boldsymbol{q} \to \partial_2 S(\boldsymbol{q}, \boldsymbol{Q}_0)$ (resp. $\boldsymbol{Q} \to \partial_1 S(\boldsymbol{q}_0, \boldsymbol{Q})$) is a diffeomorphism of the open set $\{(\boldsymbol{q}, \boldsymbol{Q}_0)\} \cap V$ (resp. $\{(\boldsymbol{q}_0, \boldsymbol{Q})\} \cap V$) of M into $\left(T^*_{\boldsymbol{Q}_0} M\right) \cap U$ (resp. $\left(T^*_{\boldsymbol{q}_0} M\right) \cap U$) for each $\boldsymbol{Q}_0$ (resp. $\boldsymbol{q}_0$.)

(ii) $S(\boldsymbol{q}_0, \boldsymbol{q}_0) = 0$, for a given $\boldsymbol{q}_0$.

This correspondence is given by:

$$F(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P}) \Leftrightarrow \begin{cases} \boldsymbol{p} = & -\partial_1 S(\boldsymbol{q}, \boldsymbol{Q})) \\ \boldsymbol{P} = & \partial_2 S(\boldsymbol{q}, \boldsymbol{Q})). \end{cases} \tag{27.2}$$

Remark 27.3 As noted before, if $\tilde{M} \cong \mathbb{R}^n$, we can choose $\tilde{U} = \tilde{M} \times \mathbb{R}^n \cong \mathbb{R}2n$ in the above definition and proposition. In this case Corollaries 26.2 and 26.4 also remain valid.

Exercise 27.4 a) Prove Proposition 27.2. Verify that, although we have written things in local coordinates, everything in Proposition 27.4 has intrinsic meaning (e.g. $\partial_1 S(\boldsymbol{q}_0, \boldsymbol{Q})$*is* an element of $T^*_{\boldsymbol{q}_0} M$, which only depends on the point $\boldsymbol{q}_0$ and not the coordinate system chosen).
b) Prove that if M in Proposition 27.2is the covering space of a manifold N with fundamental group Γ, and if S satisfy $S(\gamma \boldsymbol{q}, \gamma \boldsymbol{Q}) = S(\boldsymbol{q}, \boldsymbol{Q})$ as well as (i) and (ii), then the symplectic twist map that S generates is a lift of a symplectic twist map on N.

Exercise 27.5 Show that the set of C^1 twist maps on a compact neighborhood in the cotangent bundle of a manifold is open (*Hint*: prove first that the twist condition is an open condition).

A. The Standard Map on Hyperbolic Manifolds

The examples of symplectic twist maps in general cotangent bundles will mainly come from the next chapter, as time ε of Hamiltonian system satisfying the Legendre condition. In this section, we generalize the standard map further to cotangents of hyperbolic manifolds. We assume a little background in Riemmannian geometry, some of which we review in 26. Recall that a *hyperbolic manifold* M of dimension n is one that is covered by the hyperbolic half space $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ given the Riemmannian metric $ds^2 = \frac{1}{x_n^2} \sum_1^n dx_k^2$ (???), which has constant curvature -1. Geodesics on $\mathbb{H}^n$ are open semi circles or straight lines perpendicular to the boundary $\{x_n = 0\}$. The relevant property of the geometry of $\mathbb{H}^n$, and hence of any hyperbolic manifold, is that the exponential map is a *global* diffeomorphism $\exp : T\mathbb{H}^n \to \mathbb{H}^n \times \mathbb{H}^n$, a corollary of the Hopf-Rinow Theorem (Gallot, Hulin and Lafontaine (1987), Section ???). The generalization of the standard map that we present now is in fact valid for any Riemmanian manifold with this property.

Proposition 27.6 *Let $S : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ be given by:*

$$S(\boldsymbol{q}, \boldsymbol{Q}) = \frac{1}{2} \mathrm{Dis}^2(\boldsymbol{q}, \boldsymbol{Q}) + V(\boldsymbol{q}),$$

where $V : \mathbb{H}^n \to \mathbb{R}$ is some C^2 function, and Dis *is the distance given by the hyperbolic metric. Then S is the generating function for a symplectic twist map that we called the* generalized standard map *on $\mathbb{H}^n$. Furthermore, if V is equivariant under a group of isometries Σ of $\mathbb{H}^n$ representing the fundamental group of the hyperbolic manifold $M = \mathbb{H}^n/\Sigma$, then S is the generating function for a lift of a symplectic twist map on T^*M.*

Proof. We show that S complies with the hypothesis of Proposition 27.4. We take $M = \mathbb{H}^n, U = T^*\mathbb{H}^n \cong \mathbb{H}^n \times \mathbb{R}^n$. We now prove that $\boldsymbol{q} \to \partial_2 S(\boldsymbol{q}, \boldsymbol{Q}_0)$ (resp. $\boldsymbol{Q} \to \partial_1 S(\boldsymbol{q}_0, \boldsymbol{Q})$) is a diffeomorphism $\mathbb{H}^n \to \mathbb{R}^n$. In Section 26we remind the reader how the geodesic flow and the exponential map of a Riemmannian manifold can be seen both on the tangent bundle and the cotangent bundle (via the duality given by the Legendre transform). In the cotangent bundle the geodesic flow G^t is the hamiltonian flow with Hamiltonian the dual metric $g(\boldsymbol{q})(\boldsymbol{p}, \boldsymbol{p})$ and the exponential map is $\exp_{\boldsymbol{q}}(\boldsymbol{p}) = \pi \circ G^1(\boldsymbol{q}, \boldsymbol{p}) = \boldsymbol{Q}(\boldsymbol{q}, \boldsymbol{p})$, where $G^1(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P})$. We also prove that, if $(\boldsymbol{q}, \boldsymbol{Q})$ is in the range where $(\boldsymbol{q}, \boldsymbol{p}) \to \boldsymbol{q} \times \exp(\boldsymbol{q}, \boldsymbol{p})$ has an inverse (the case for all $(\boldsymbol{q}, \boldsymbol{Q}) \in \mathbb{H}^n \times \mathbb{H}^n$ here), then:

$$(27.3)\qquad \begin{aligned} \partial_1 \mathrm{Dis}(\boldsymbol{q}, \boldsymbol{Q}) &= \frac{-\boldsymbol{p}}{\|\boldsymbol{p}\|} = \frac{-\boldsymbol{p}}{\mathrm{Dis}(\boldsymbol{q}, \boldsymbol{Q})} \\ \partial_2 \mathrm{Dis}(\boldsymbol{q}, \boldsymbol{Q}) &= \frac{\boldsymbol{P}}{\|\boldsymbol{P}\|} = \frac{\boldsymbol{P}}{\mathrm{Dis}(\boldsymbol{q}, \boldsymbol{Q})} \end{aligned}$$

and hence $\partial_1 \mathrm{Dis}^2(\boldsymbol{q}, \boldsymbol{p}) = -\boldsymbol{p}, \partial_2 \mathrm{Dis}^2(\boldsymbol{q}, \boldsymbol{p}) = \boldsymbol{P}$. The assumption that the exponential is a diffeomorphism means, in this notation, that $\boldsymbol{p} \to \boldsymbol{Q}(\boldsymbol{q}_0, \boldsymbol{p})$ is a diffeomorphism for each fixed $\boldsymbol{q}_0$ and G^1 is a symplectic twist map . Likewise $\boldsymbol{P} \to \boldsymbol{q}(\boldsymbol{Q}_0, \boldsymbol{P})$ is a diffeomorphism because G^{-1}, the inverse of asymplectic twist map must be a symplectic twist map itself. Thus we have established that the maps $\boldsymbol{q} \mapsto \partial_2 \frac{1}{2}\mathrm{Dis}^2(\boldsymbol{q}, \boldsymbol{Q}_0)$ and $\boldsymbol{Q} \mapsto \partial_1 \frac{1}{2}\mathrm{Dis}^2(\boldsymbol{q}_0, \boldsymbol{Q})$ are both diffeomorphisms for each *fixed* $\boldsymbol{q}_0, \boldsymbol{Q}_0$. Coming back to our generating function, we have proven that:

$$\boldsymbol{q} \mapsto \partial_2 S(\boldsymbol{q}, \boldsymbol{Q}_0) = \partial_2 \frac{1}{2}\mathrm{Dis}^2(\boldsymbol{q}, \boldsymbol{Q}_0)$$

is a diffeomorphism, and

$$\boldsymbol{Q} \mapsto \partial_1 S(\boldsymbol{q}_0, \boldsymbol{Q}) = \partial_1 \frac{1}{2}\mathrm{Dis}^2(\boldsymbol{q}_0, \boldsymbol{Q}) + dV(\boldsymbol{q}_0)$$

must also be a diffeomorphism $\mathbb{H}^n \to T_{\boldsymbol{q}_0}\mathbb{H}^n$ since we added a constant translation by $dV(\boldsymbol{q}_0)$ to a diffeomorphism. Proposition 27.4concludes the proof that S is the generating function for a twist map of $T^*\mathbb{H}^n$. The last statement of the proposition is an easy consequence of Exercise 27.4. □

28. Elliptic Fixed Points

As we will see in Appendix 1 or SG, the study of Hamiltonian dynamics around a periodic orbit of a time independent Hamiltonian reduces to that of a symplectic map:

$$\mathcal{R} : \mathbb{R}2n \to \mathbb{R}2n, \quad \text{such that} \quad \mathcal{R}(0) = 0,$$

called the Poincaré return map.

We now follow Moser (1977). If 0 is an elliptic fixed point, that is $D\mathcal{R}(0)$ has all its eigenvalues on the unit circle, a normal form theorem ???(find ref.) says that (generically?) the map $\mathcal{R}$ is, around 0 given by:

$$Q_k = q_k cos\Phi_k(\boldsymbol{q},\boldsymbol{p}) - p_k sin\Phi_k(\boldsymbol{q},\boldsymbol{p}) + f_k(\boldsymbol{q},\boldsymbol{p})$$
$$P_k = q_k sin\Phi_k(\boldsymbol{q},\boldsymbol{p}) + p_k cos\Phi_k(\boldsymbol{q},\boldsymbol{p}) + g_k(\boldsymbol{q},\boldsymbol{p})$$
$$\Phi_k(\boldsymbol{q},\boldsymbol{p}) = \alpha_k + \sum_{l=1}^{n} \beta_{kl}(q_l^2 + p_l^2).$$

where the error term f_k, g_k are C^3.[5]

We now show how this map is, in "polar coordinates", a symplectic twist map of $T^*\mathbb{T}^n$, *whenever the matrix $\{\beta_{kl}\}$ is non singular.*

Let V be a punctured neighborhood of 0 such that: $0 < \sum_k (q_k^2 + p_k^2) < \epsilon$.

We introduce on V new coordinates (r_k, θ_k) by:

$$q_k = \sqrt{2r_k\epsilon} cos 2\pi\theta_k, \quad p_k = \sqrt{2r_k\epsilon} sin 2\pi\theta_k$$

where θ_k is determined modulo 1. One can check that V is transformed into the "annular" set:

$$U = \left\{ (\theta_k, r_k) \in Tn \times \mathbb{R}^n \quad | \quad r_k > 0 \quad \text{and} \quad \sum_k r_k < \frac{1}{2} \right\}$$

Since the symplectic form $d\boldsymbol{q} \wedge d\boldsymbol{p}$ is transformed into $2\pi\epsilon d\boldsymbol{r} \wedge d\boldsymbol{\theta}$, $\mathcal{R}$ remains symplectic in these new coordinates, with the symplectic form $d\boldsymbol{r} \wedge d\boldsymbol{\theta}$. In fact, $\mathcal{R}$ is exact symplectic in U. To check this, it is enough to show that, for any closed curve γ:

$$\int_{\mathcal{R}\gamma} \boldsymbol{r} d\boldsymbol{\theta} = \int_{\gamma} \boldsymbol{r} d\boldsymbol{\theta}.$$

(see Exercise 46.7). It is easy to see that $4\pi\epsilon r_k d\theta_k = p_k dq_k - q_k dp_k$, so by Stokes' theorem:

$$4\pi\epsilon \int_{\gamma} \boldsymbol{r} d\boldsymbol{\theta} = \int_{\partial D} \boldsymbol{p} d\boldsymbol{q} - \boldsymbol{q} d\boldsymbol{p} = -2 \int_D \omega$$

where D is a 2 manifold in V with boundary $\partial D = \gamma$. Since $\mathcal{R}$ preserves ω in V, it must preserve the last integral, and hence the first.

To see that $\mathcal{R}$ satisfies the two other conditions for being a symplectic twist map, we just write $\mathcal{R}(\boldsymbol{\theta}, \boldsymbol{r}) = (\boldsymbol{\Theta}, \boldsymbol{R})$ in the new coordinates then:

$$\Theta_k = \theta_k + \psi_{F_k}(\boldsymbol{r}) + o_1(\epsilon)$$
$$R_k = r_k + o_1(\epsilon)$$
$$\text{with } \psi_{F_k} = \alpha_k + \epsilon \sum_{l=1}^{n} 2\beta_{kl} r_l.$$

where $\epsilon^{-1} o_1(\epsilon, \boldsymbol{\theta}, \boldsymbol{r})$ and its first derivatives in $\boldsymbol{r}, \boldsymbol{\theta}$ tend to 0 uniformally as $\epsilon \to 0$. We can rewrite this as:

$$\mathcal{R}(\boldsymbol{\theta}, \boldsymbol{r}) = (\boldsymbol{\theta} + \epsilon \boldsymbol{B}\boldsymbol{r} + \boldsymbol{\alpha} + o_1(\epsilon), \boldsymbol{r} + o_1(\epsilon)).$$

[5] actually, one only need them to have vanishing derivatives up to order 3 at the origin and be C^1 otherwise.

So for small ϵ, the condition $\det \partial\boldsymbol{\Theta}/\partial\boldsymbol{r} \neq 0$ is given by the nondegeneracy of $\boldsymbol{B} = \{\beta_{kl}\}$, one uses the fact that $\mathcal{R}$ is C^1 close to a completely integrable symplectic twist map to show that $\mathcal{R}$ is twist in U (the twist condition is open.) The fact that it is homotopic to Id derives from Exercise 23.2.

Note that the set V and therefore U are not invariant under $\mathcal{R}$. However, it is still possible to show the existence of infinitely many periodic points for $\mathcal{R}$: this is the content of the Birkhoff– Lewis theorem (???: state it precisely somewhere) (see Moser (1977)) .

Lemma STMdiffeo is 26.3, Exercise STMstmopen is 27.5, example STMstandardexample is 25.1 , Proposition STMsuffstm is 26.2, formerly a Corollary (Coro), Proposition STMlocglobal is 26.4, Section STMsecelliptic is 28.0, Proposition STMpropdiff is 26.5, STMpropactionpr is Proposition 24.1, Exercise STMexohomt is 23.2.