# 2 or AM

# THE AUBRY-MATHER THEOREM

June 15 1999

**I have decided to make a compact chapter, with no flow proof. That will come in the new GCchapter.**

**Action to be taken: Find the proof for the lemma extending annulus maps to cylinder maps. Find the drawing of Chenciner for Lipshitz condition on AM sets. Draw the 4 figures. Find the right reference for the no crossing lemma in refmanemml. Add a statement of KAM before AM. Proofread**

## 8. Introduction

The orbits of the twist map $f_0$ whose lift is the completely integrable shear map given by $F_0(x, y) = (x+y, y)$, possess the following four fundamental properties, some of which we have yet to define:

(1) They lie on invariant circles which are graphs over the circle $\{y = 0\}$.

(2) They are ordered cyclically, as orbits of rotations on the circle.

(3) They come with all rotation numbers in $(-\infty, +\infty)$.

(4) They are action minimizers.

The KAM theorem (see THMkam) implies that, in the measure sense, most of these invariant circles will "survive" a *small* perturbation of $f_0$. The rotation number of these survivors has to be very irrational (diophantine). One cannot hope for all these circles to survive under arbitrary perturbation of the map $f_0$. In fact, it is known (ref ???: check jdm) that for $k > 0.9716354$, the standard map has no invariant circle. In the context of the Standard family, the Aubry-Mather theorem implies that, for each invariant circle of $f_0$, and for each $\lambda > 0$, there exists an invariant set for $f_\lambda$ which can be seen as the remnant of the invariant circle. The properties of the orbits exhibited by the Aubry-Mather theorem will all be defined in subsequent sections.

**Theorem 8.1 (Aubry-Mather)** *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the lift of a $C^2$ twist map of the cylinder with generating function $S$ satisfying the following growth condition:*

$$\lim_{|X-x|\to\infty} S(x, X) \to +\infty$$

*Then $F$ has orbits of all rotation numbers in $\mathbb{R}$ Moreover, these orbits can be chosen to have the following properties:*

*(1) They are cyclically ordered*

*(2) They lie on closed $F$-invariant sets, called* Aubry- Mather sets *that form graphs over their projection on the circle $\{y = 0\}$ and are conjugated to closed invariant sets of lifts of circle homeomorphisms: either lifts of periodic orbits, Denjoy Cantor sets (and optionally, orbits homoclinic to these sets) or the full circle.*

*(3) They may be chosen to be action minimizers.*

We will see that an invariant Cantor sets must occur each time there is no invariant circle of a given irrational rotation number. The existence of these invariant Cantor sets was the striking novelty of this theorem. Often, the term "Aubry-Mather sets" is restricted to denote only the invariant Cantor sets.

#### Sketch of the proof

We will find periodic orbits of all rational rotation numbers by minimizing the periodic action $W_{mn}$. Aubry's Fundamental Lemma will imply that $W_{mn}$-minimizers are "cyclically ordered", *i.e.* ordered like orbits of circle homeomorphisms. The cyclic order property enables us to take limits of these periodic orbits (they will be in a compact set if their rotation numbers are in a bounded set). Cyclic order also implies that the rotation number of the limiting orbit exists and is the limit of the rotation numbers of the periodic orbits.

One way in which this presentation differs from the excellent surveys of this subject by Meiss (1992) or Hasselblat & Katok (1995) is the focus on the cyclic order property at the level of sequences (that are not necessarily realized by orbits). I found it a convenient bridge between the study of the dynamics of circle homeomorphisms (which appears in the appendix to this chapter) and that of Aubry-Mather sets.

We preceed our study by a Lemma, which implies that we can reduce our study to twist maps of the cylinder.

**Lemma 8.2** *Let $f$ be a $C^k, k \geq 2$, twist map of a compact annulus $\mathcal{A}$. Then $f$ can be extended to a $C^k$ twist map of the cylinder $\mathcal{C}$, in such a way that it coincides with the shear map $(x, y) \mapsto (x + cy, y)$ outside a compact set. In particular, letting $k \geq 2$ the generating function of any lift of the extended map satisfies the growth condition $\lim_{|X-x|\to\infty} S(x, X) \to +\infty$.*

As a corollary of this lemma, we can specialize the Aubry-Mather theorem to maps of the compact annulus:

**Theorem 8.3 (Aubry-Mather on the compact annulus)** *Let $F$ be the lift of a twist map of the bounded annulus and suppose that the rotation numbers of the restriction of $F$ to the lower and upper boundaries are $\rho_-$, and $\rho_+$ respectively. Then $F$ has orbits of all rotation numbers in $[\rho_-, \rho_+]$. These orbits are minimizers, recurrent, cyclically ordered and they lie on compact invariant sets that form (uniformly) Lipshitz graphs over their projections. These sets may either be periodic orbits, invariant circles or invariant Cantor sets on which the map is semi-conjugate to lifts of circle rotations.*

*Proof.* ???

## 9. Cyclically Ordered Sequences and Orbits

If a map $G : \mathbb{R} \to \mathbb{R}$ is the lift of a circle homeomorphism which preserves the orientation, it is necessarily strictly increasing and must satisfy $G(x + 1) = G(x) + 1$. Hence, if $\{x_k\}_{k\in\mathbb{Z}}$ is an orbit of $G$, it must satisfy:

$$x_k \leq x_j + p \Rightarrow x_{k+1} \leq x_{j+1} + p, \forall k, j, p \in \mathbb{Z}. \tag{9.1}$$

We will say that a sequence $\{x_k\}_{k\in\mathbb{Z}}$ in $\mathbb{R}^{\mathbb{Z}}$ is *Cyclically Ordered*, (or *CO* in short) if it satisfies (9.1) . Clearly the CO sequences form a closed set for the *topology of pointwise convergence* in $\mathbb{R}^{\mathbb{Z}}$: $\boldsymbol{x}^{(j)} \to \boldsymbol{x}$ whenever $x_k^j \to x_k$ for all $k$. Not that this topology is the same as the product topology on the space of sequences. Using the *partial order on sequences*

$$\boldsymbol{x} < \boldsymbol{y} \Leftrightarrow \{x_k \leq y_k, \boldsymbol{x} \neq \boldsymbol{y}\},$$

we let the reader check that an equivalent definition of CO sequences is:

$$\forall m, n \in \mathbb{Z}, \qquad \tau_{m,n}\boldsymbol{x} \geq \boldsymbol{x} \quad \text{or} \quad \tau_{m,n}\boldsymbol{x} \leq \boldsymbol{x} \tag{9.2}$$

where

$$(\tau_{m,n}\boldsymbol{x})_k = x_{k+m} + n.$$

We will investigate this order relation and the maps $\tau_{m,n}$ in greater detail in GCchapter. We say that the orbit $\{(x_k, y_k)\}_{k\in\mathbb{Z}}$ of a twist map is a *Cyclically Ordered orbit* or *CO orbit* if $\{x_k\}_{k\in\mathbb{Z}}$ is CO. These orbits come with various other names in the litterature: *Well Ordered* (has no hint of the cyclic ordering), *Monotone* (is used in too many contexts), *Birkhoff* (this order was apparently never mentioned by Birkhoff) [3]

[3] This is not an indictment of the authors who have used these terminologies: the author of this book has himself used them all in various publications...

**Lemma 9.1** *Let* $\{x_k\}_{k\in\mathbb{Z}}$ *be a CO sequence then* $\rho(\boldsymbol{x}) = \lim_{k\to\infty} x_k/k$ *exists and:*

$$|x_k - x_0 - k\rho(\boldsymbol{x})| \leq 1. \tag{9.3}$$

*Moreover* $\boldsymbol{x} \to \rho(\boldsymbol{x})$ *is a continuous function on CO sequences, when the set of sequences has been given the topology of pointwise convergence.*

Define:

$$CO_{[a,b]} = \{\boldsymbol{x} \in CO \mid \rho(\boldsymbol{x}) \in [a, b]\}.$$

The following lemma shows that it is easy to find limits of $CO$ sequences, as long as their rotation numbers are bounded.

**Lemma 9.2** *The sets* $CO_{[a,b]}/\tau_{1,0}$ *and* $CO_{[a,b]} \cap \{\boldsymbol{x} \in \mathbb{R}^{\mathbb{Z}} \mid x_0 \in [0,1]\}$ *are compact for the topology of pointwise convergence.*

We give the (simple) proofs of both these lemmas in the appendix to this chapter. The fact, given by these lemmas, that the rotation number behaves well under limits of CO-sequences is one of the essential points in the theory of twist maps that does not generalize to higher dimensional maps: to our knowledge, there is no canonical definition of CO sequences in $\mathbb{R}^n, n \geq 2$ which ensures the existence of rotation vectors which behave well under limits.

There is a visual way to describe CO sequences, which we now come to. A sequence $\boldsymbol{x}$ in $\mathbb{R}^{\mathbb{Z}}$ is a function $\mathbb{Z} \to \mathbb{R}$. One can interpolate this function linearly to give a piecewise affine function $\mathbb{R} \to \mathbb{R}$ that we denote by $t \mapsto x_t$. The graph of this function is sometimes called the *Aubry diagram* of the sequence. We say that two sequences $\boldsymbol{x}$ and $\boldsymbol{w}$ *cross* if their corresponding Aubry diagrams cross. There are two types of crossing: at an integer $k$, in which case $(x_{k-1} - w_{k-1})(x_{k+1} - w_{k+1}) < 0$ or at a non integer $t \in (k, k+1)$, in which case $(x_k - w_k)(x_{k+1} - w_{k+1}) < 0$. These inequalities can be taken as a definition of crossings. Non–crossing of two sequences can be put in terms of the strict partial order on sequence: $\boldsymbol{x}, \boldsymbol{y}$ do not cross if and only if $\boldsymbol{x} < \boldsymbol{y}$ . *In particular a sequence* $\boldsymbol{x}$ *is CO if and only if it has no crossing with any of its translates* $\tau_{m,n}\boldsymbol{x}$.

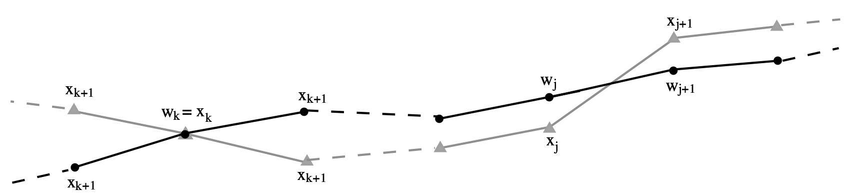


**Fig. 9. 0.** Aubry diagrams of sequences and their crossings: in this example the sequences $\boldsymbol{x}$ and $\boldsymbol{w}$ have crossings at the integer $k$ and between the integers $j$ and $j+1$.

## 10. Minimizing Orbits

A sequence segment $(x_k, \ldots, x_m)$ is *(action) minimizing* if

$$W(x_k, \ldots, x_m) \leq W(y_k, \ldots, y_m)$$

for any other sequence segment $(y_k, \ldots, y_m)$ with same endpoints: $x_k = y_k, x_m = y_m$. Since minimizing segments are necessarily critical for $W$, they correspond to orbit segments called *(action) minimizing orbit segment* . A bi-infinite sequence is called a *(global action) minimizer* if any of its segments is minimizing and the orbit it corresponds to is a *minimizing orbit*, also called minimizer, when the context is clear. Note that the set of minimizers is closed under the topology of pointwise limit. Finally a $W_{mn}$*-minimizers* in $X_{mm}$, is a periodic sequences that minimize the function $W_{mn}$.

A recurrent theme in the Calculus of Variation is that minimizers have regimented crossings. In the case of geodesics on a Riemmanian manifold, geodesics that (locally) minimize length cannot have conjugate points, *i.e.* small variations with fixed endpoints of a minimizing geodesic only intersect that geodesic at the endpoints, ( Milnor (1969)), and geodesics that minimize length globally cannot have self intersections (Manẽ (1991)). We will see, in the present theory, that minimizers satisfy a non-crossing condition, which implies that $W_{mn}$–minimizers are CO (and more generally that recurrent minimizers are CO).

**Lemma 10.1 (crossing)** *Suppose that* $(x - w)(X - W) \leq 0$. *Then:*

$$S(x, X) + S(w, W) - S(x, W) - S(w, X) \leq 0,$$

*and equality occurs iff* $(x - w)(X - W) = 0$

*Proof*. We can write:

$$S(x, X) - S(x, W) = \int_0^1 \partial_2 S(x, X_s)(X - W)ds,$$

where $X_s = (1 - s)W + sX$. Applying the same process to $h(x) = S(x, X) - S(x, W)$, we get:

$$S(x, X) + S(w, W) - S(x, W) - S(w, X) = h(x) - h(w) =$$
$$-\int_0^1 \int_0^1 \partial_{12} S(x_r, X_s)(X - W)(x - w)dsdr = \lambda(X - W)(x - w)$$

for some strictly negative $\lambda$, by the positive twist condition and for $x_r = (1 - r)w + rx$. □

The following is a watered down version of the Fundamental Lemma in Aubry & Le Daeron (1983). We follow Meiss (1992) :

**Lemma 10.2 (Aubry's Fundamental Lemma)** *Two distinct minimizers cross at most once.*

*Proof*. Suppose that $\boldsymbol{x}$ and $\boldsymbol{w}$ are two minimizers who cross twice. We perform some surgery on finite segments of $\boldsymbol{x}$ and $\boldsymbol{w}$ to get two new sequences $\boldsymbol{x}'$ and $\boldsymbol{w}'$ with at least one of them of lesser action, contradicting minimality. There are three cases to consider: (i) both crossings are at non integers, (ii) one crossing is at an integer, (iii) both crossings are at integers.

*Case (i):* Let $t_1 \in (i-1, i)$ and $t_2 \in (j, j+1)$ be the crossing times. Define:

$$x'_k = \begin{cases} w_k & \text{if} \quad k \in [i,j] \\ x_k & \text{otherwise} \end{cases} \qquad w'_k = \begin{cases} x_k & \text{if} \quad k \in [i,j] \\ w_k & \text{otherwise} \end{cases}$$

Letting $W$ denote the action over an interval $[N, M]$ containing $[j-1, k+1]$, we easily compute that:

$$\begin{aligned} W(\boldsymbol{x}') + W(\boldsymbol{w}') - W(\boldsymbol{x}) - W(\boldsymbol{w}) = \\ S(x_{i-1}, w_i) + S(w_{i-1}, x_i) - S(x_{i-1}, x_i) - S(w_{i-1}, w_i) \\ + S(x_j, w_{j+1}) + S(w_j, x_{j+1}) - S(x_j, x_{j+1}) - S(w_j, w_{j+1}). \end{aligned}$$

The Crossing Lemma 10.1shows that this difference of actions is negative, contradicting the minimality of $\boldsymbol{x}$ and $\boldsymbol{w}$.

*Case (ii):* In this case, only one crossing will contribute negatively to the difference of action of new and old sequences. We still get a contradiction.

*Case (iii)* Let $i-1$ and $j+1$ be the crossing times $\boldsymbol{x}$ amd $\boldsymbol{w}$, and construct $\boldsymbol{x}'$ and $\boldsymbol{w}'$ as before. In this case the difference in action between old and new segments is null. All the sequences must be minimizing, and hence correspond to orbits. But we have $x_{i-2} = w'_{i-2}, \quad x_{i-1} = w'_{i-1}$. Hence the points $\psi^{-1}(x_{i-2}, x_{i-1})$ and $\psi^{-1}(w'_{i-2}, w'_{i-1})$ of $\mathbb{R}^2$ are the same and generate a unique orbit under $F$. This in turn implies that $\boldsymbol{x} = \boldsymbol{w}$ are not distinct. □

**Corollary 10.3** *$W_{mn}$-minimizing sequences are CO and their set is completely ordered for the partial order on sequences.*

*Proof.* Since the proof of Aubry's Lemma dealt with finite segments of sequences only, it also applies to show that two $W_{mn}$-*minimizers* in $X_{mm}$, may not cross twice within one period $n$. But two $m, n$-periodic sequences that cross once must necessarily cross twice within one period. Hence two $W_{mn}$-minimizers cannot cross at all. If $\boldsymbol{x}$ is a $W_{mn}$ minimizer, $\tau_{i,j}\boldsymbol{x}$ is also a $W_{mn}$-minimizer. Since they do not cross, one must have either $\boldsymbol{x} < \tau_{i,j}\boldsymbol{x}$ or $\tau_{i,j}\boldsymbol{x} < \boldsymbol{x}$, for all $i, j \in \mathbb{Z}$, *i.e.* $\boldsymbol{x}$ is CO. □

We end this section by a proposition which we will need only in GCchapter.

**Proposition 10.4** *Any $W_{mn}$-minimizer is a minimizer.*

*Proof.* We show that if $\boldsymbol{x}$ is a $W_{mn}$-minimizer is also a $W_{kmkn}$ minimizer for any $k$. This implies that $\boldsymbol{x}$ is a minimizer on segments of arbitrary length: if $\boldsymbol{x}$ is a $W_{kmkn}$ minimizer, any segment of $\boldsymbol{x}$ of length less than $kn$ is minimizing. Hence $\boldsymbol{x}$ is a minimizer. Take a $W_{kmkn}$-minimizer $\boldsymbol{w}$. If $\boldsymbol{w}$ is not $m, n$-periodic, then $\boldsymbol{w}$ and $\tau_{m,n}\boldsymbol{w}$ are distinct. By the Corollary 10.3, they cannot cross. Suppose, say, that $\tau_{m,n}\boldsymbol{w} > \boldsymbol{w}$. Since $\tau_{m,n}$ trivially preserves the order on sequences, we must also have $\tau^k_{m,n}\boldsymbol{w} > \boldsymbol{w}$, a contradiction to the fact that $\boldsymbol{w}$ is $km, kn$- periodic. Hence $\boldsymbol{w}$ is in $X_{mn}$ and its action over intervals of any length multiple of $n$ cannot be less than that of $\boldsymbol{x}$. Hence $\boldsymbol{x}$ is also a $W_{kmkn}$ minimizer. □

**Exercise 10.5** Show that a minimizer corresponding to a recurrent (not necessarily periodic) orbit of the twist map is CO. (Remember that the orbit $z_k$ of a dynamical system is called *recurrent* if $z_0$ is the limit of a subsequence $z_{k_j}$. Equivalently, $z_0$ is in its own $\omega$–limit set). More generally, show that the set of minimizers of rotation number $\omega$ is completely ordered. (*Hint.* Mimic the proof of Proposition 10.4 : if an appropriate inequality is not satisfied, there must be a crossing. By recurrence, there is another one, a contradiction to Aubry's Lemma).

## 11. CO Orbits of All Rotation Numbers

### A. CO Periodic Orbits

We prove that the set of $W_{mn}$-minimizers is not empty. By Corollary 10.3 this will show the existence of $CO$ orbits of all rational rotation numbers.

**Proposition 11.1** *Let the twist condition for the lift of a twist map $F$ be uniform:*

$$\frac{\partial X(x,y)}{\partial y} > a > 0 \quad \forall (x,y) \in \mathbb{R}^2.$$

*Then $W_{mn}$ is proper and bounded below, and hence has a minimum.*

We remind the reader that $h : X \to \mathbb{R}$ is *proper function* if the inverse image of a compact set is compact. If $X = \mathbb{R}^n$, then this translates to: the inverse image of any bounded interval is bounded. If $h$ is also bounded below, it must indeed attain the $\inf_{x\in\mathbb{R}^n} = \alpha$ for some $x_0$ since, for instance, $h^{-1}[\alpha-1, \alpha+1]$ is compact.

*Proof of Proposition 11.1* It is an immediate consequence of the following lemma (see MacKay & al. (1989)):

**Lemma 11.2** *There is a constant $\alpha$, and two strictly positive constants $\beta$ and $\gamma$ such that :*

$$S(x,X) \geq \alpha - \beta\,|X-x| + \gamma\,|X-x|^2$$

*Proof.* We can write:

$$S(x,X) = S(x,x) + \int_0^1 \partial_2 S(x, X_s)(X-x)ds,$$

where $X_s = (1-s)x + sX$. Applying the same process to $\partial_2 S$, we get:

$$\begin{aligned} S(x,X) = S(x,x) + \int_0^1 \partial_2 S(X_s, X_s)(X-x)ds \\ - \int_0^1 ds \int_0^1 \partial_{12} S(X_r, X_s)(X-x)^2 dr \end{aligned}.$$

We can conclude the proof of the lemma by taking

$$\alpha = \min_{x\in\mathbb{R}} S(x,x), \quad \beta = \max_{x\in\mathbb{R}} |\partial_2 S(x,X)|$$

(which exist by periodicity of $S$) and $\gamma = a/2$. $\square$

### B. CO Orbits of Irrational Rotation Numbers

The existence of CO orbits of irrational rotation numbers is a simple consequence of the existence of CO periodic orbits: pick a sequence $\boldsymbol{x}^{(k)}$ of $W_{m_k,n_k}$–minimizers, with $m_k/n_k \to \omega$ as $k \to \infty$. By using appropriate translations of the type $\tau_{m,0}$ on $\boldsymbol{x}^{(k)}$ (which neither change their rotation numbers, nor the fact that they are minimizers) we can assume that $\boldsymbol{x}^{(k)} \in [0,1]$. The sequence $m_k/n_k$ is bounded and hence, by Corollary 10.3 the sequences $\boldsymbol{x}^{(k)}$ are in $CO_{[a,b]} \cap \{\boldsymbol{x} \in \mathbb{R}^{\mathbb{Z}} \mid x_0 \in [0,1]\}$ for some $a, b \in \mathbb{R}$. Lemma 9.2 garantees the existence of a converging subsequence in $CO_{[a,b]}$ and Lemma 9.1 shows that the limit of this subsequence has rotation number $\omega$. Finally, note that the periods $n_k \to \infty$ as $k \to \infty$. In particular, any finite segment of $\boldsymbol{x}$ is the limit of minimizing segments, hence minimizing itself. □

## 12. Aubry-Mather Sets

We have proven Part (1) and (3) of the Aubry-Mather theorem: existence of cyclically ordered, minimizing orbits of all rotation numbers. We now prove Part (2) of the Aubry-Mather theorem: the cyclically ordered orbits that we found in the previous section lie on Aubry-Mather sets, which now describe.

We say that a set $M$ in $\mathbb{R}^2$ is $F$-*ordered* if, for $z, z'$ in $M$,

$$\pi(z) < \pi(z') \Rightarrow \pi(F(z)) < \pi(F(z')),$$

where $\pi$ is the $x$-projection. If moreover $M$ is invariant by $F$ and $F^{-1}$, then the sequences $\boldsymbol{x}, \boldsymbol{x}'$ of $x$–coordinates of $z$ and $z'$ satisfy $\boldsymbol{x} \prec \boldsymbol{x}'$. An example of $F$-ordered invariant set is the set of points in a CO orbit and all their integer translates (In fact, this is an alternative definition of CO orbits). Note that an invariant circle for the map which is a graph (we will see in INVchapterthat all invariant circles are graphs) is $F$-ordered. We now want to explore the properties of $F$-ordered invariant sets. Crucial to the properties of these sets is the following *ratchet phenomenon* (I owe this terminology to G.R. Hall), which is a somewhat quantitative expression of the twist condition:

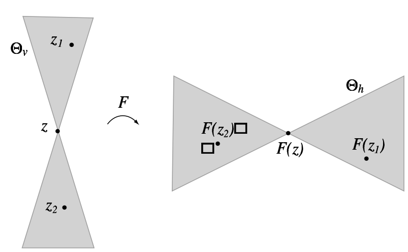


**Fig. 12. 0.** The ratchet phenomenon for the lift of a positive twist map $F$: there are two cones $\Theta_v$ and $\Theta_h$ in $\mathbb{R}^2$ centered around the $y$ and $x$-axes respectively, such that, if $z, z'$ are two points of $\mathbb{R}^2$ with $z' \in z + \Theta_v$, then $F(z') \in F(z) + \Theta_h$. More precisely, for a positive twist map $z' \in z + \Theta_v^+ \Rightarrow F(z') \in F(z) + \Theta_h^+$, where the half cones $\Theta_h^+, \Theta_v^+$ have the obvious meaning. The same holds for the half cones $\Theta_h^-$ and $\Theta_v^-$. If $g$ is *negative* twist (*eg.* $F^{-1}$), then the signs are reversed. The same cones can be used for $F^{-1}$ as for $F$.

**Lemma 12.1** *Let $F$ be the lift of a twist map satisfying $\frac{\partial X}{\partial y} > a > 0$ in some region. Then, in that region, $F$ satisfies the ratchet phenomenon for some cones $\Theta_v, \Theta_h$ whose angles only depend on $a$.*

*Proof.* Left as an exercise.

**Proposition 12.2** *The closure of an $F$–ordered invariant set is $F$–ordered and invariant.*

*Proof.* The invariance is by continuity of $F$. Suppose that, in the closure $\overline{M}$ of $M$ there are $z, z'$ in $\overline{M}$, with $\pi(z) < \pi(z')$ but $\pi(F(z)) = \pi(F(z'))$ (the worst scenario). By the ratchet phenomenon for $F^{-1}$, $F(z)$ must be above $F(z')$ and $\pi(F^2(z')) < \pi(F^2(z))$, *i.e.* the $x$ orbits of $z$ and $z'$ switched order. This is impossible since in $M$ the order is preserved.

**Proposition 12.3** *If $M$ is an $F$-ordered invariant set, then it is a Lipschitz graph over its projection: there exists a constant $K$ depending only on $F$ such that, if $(x, y)$ and $(x\prime, y\prime)$ are two points of $M$, then:*

$$|y\prime - y| \leq K|x\prime - x|$$

*with $K$ only depending on the twist constant $a = \inf_M \frac{\partial X}{\partial y}$.*

Note that $a$, and hence $K$ can be chosen the same for all $F$-ordered sets in a compact region.

**Remark 12.4** Applied to the special case of invariant circles, Proposition 12.3 shows that any invariant circle for a twist map which is a graph is Lipschitz. This is a theorem originally due to Birkhoff, who also proved (see INVchapter) that all invariant circles for twist maps must be graphs.

*Proof.* The proof of Lemma 12.3 shows that if $M$ is $F$-ordered, we cannot have $z, z'$ in $M$ and $\pi(z) = \pi(z')$ unless $z = z'$. Hence $\pi$ is injective on $M$, and $M$ is a graph. To show that $M$ forms a *Lipschitz* graph over its projection, let $z$ and $z'$ be two points of $M$ and $\boldsymbol{x}$ and $\boldsymbol{x}'$ the corresponding sequences of $x$-coordinates of their orbits. Assuming $\pi(z) < \pi(z')$, we must have $\boldsymbol{x} \prec \boldsymbol{x}'$. If $z' \in z + \Theta_v^+$, the ratchet phenomenon implies that $F^{-1}(z') \in F^{-1}(z) + \Theta_h^-$, *i.e.* $x'_{-1} > x_{-1}$, a contradiction. Likewise $z'$ cannot be in the cone $z + \Theta_v^-$, and hence it must be in the cone complementary to $\Theta_v$ at $z$. This cone condition is easily transcribed into a uniform Lipschitz condition $|y' - y| < K|x' - x|$. □

**Lemma 12.5** *All points in an $F$-ordered set have the same rotation number.*

*Proof.* This is a consequence of the fact (Lemma AMlemmax<yrot in the appendix) that if $\boldsymbol{x} < \boldsymbol{x}'$ are two CO sequences, they must have the same rotation number. □

**Definition 12.6** An *Aubry-Mather set* $M$ for the lift $F$ of a twist map $f$ of the cylinder is a closed, $F$-ordered set invariant under $F, F^{-1}$ and the integer translation $T$.

**Theorem 12.7 (Properties of Aubry-Mather sets)** *Let $M$ be an Aubry-Mather set for a lift $F$ of a twist map of the cylinder.*

*(a) $M$ forms a graph over its projection $\pi(M)$, which is Lipschitz with Lipschitz constant only depending on a where $\frac{\partial X}{\partial y > a}$.*

*(b) All the orbits in $M$ are cyclically ordered and they all have the same rotation number, which is called the rotation number of $M$.*

*(c) The projection $\pi(M)$ is a closed invariant set for the lift of a circle homeomorphism, and hence $F$ restricted to $M$ is conjugated to the lift of a circle homeomorphism via $\pi$.*

We remind the reader that a *conjugacy* between two maps $F : M \to M$ and $G : N \to N$ is a homeomorphism $h : M \to N$ such that $h \circ F = H \circ h$. Taking the closure of all the integer translates of the points in the CO orbits found in the previous section, we immediately get:

**Theorem 12.8** *Let $F$ be the lift of a twist map of the cylinder. Then $F$ has Aubry-Mather sets of all rotation numbers in $\mathbb{R}$ . Any CO orbit is in an Aubry-Mather set.*

Note that this theorem gives part (b) of the Aubry-Mather theorem.

*Proof of Theorem 12.7* We have shown in Lemmas 12.6 and 12.5 that (a) and (b) are in fact properties of invariant $F$-ordered sets. As for Property (c), since $\pi$ is one to one on $M$, $F$ induces a continuous (Lipschitz, in fact) increasing map $G$ on $\pi(M)$ (by $G(\pi(z)) = \pi(F(z))$. Since $M$ and thus $\pi(M)$ are invariant under integer translation, we have $G(x + 1) = G(x) + 1$. The set $\pi(M)$ is closed and invariant under integer translation since $M$ is. If $\pi(M) = \mathbb{R}$ then $G$ is the lift of a circle homeomorphism. If $\pi(M) \neq \mathbb{R}$, then its complement is made of open intervals. The fact that $G$ is increasing on $\pi(M)$ allows one to extend $G$ by linear interpolation on each interval in the complement of $\pi(M)$. The resulting map $G$ is increasing, continuous and $G(x + 1) = G(x) + 1$, hence the lift of a circle homeomorphism. By construction $G(\pi(z)) = \pi(F(z))$, and $\pi\big|_M$ is a continuous, 1-1 map on the compact set $M$, hence a homeomorphism $M \to \pi(M)$. Thus $\pi$ is a conjugacy between $F$ on $M$ and $G$ on $\pi(M)$, which is a closed and invariant set under $G$ and $G^{-1}$. $\square$

If $G$ is the lift of a circle homeomorphism constructed in the proof of Theorem 12.7, the possible dynamics for invariant sets of circle maps described in the appendix become, under the conjugacy, possible dynamics on Aubry-Mather sets $M$ for $F$. Hence an Aubry-Mather set $M$ is either:

(i) an ordered collection of periodic orbits with (possibly) heteroclinic orbits joining them, or

(ii) the lift of an $f$-invariant circle, or

(iii) an $F$-invariant Cantor set with (possibly) homoclinic orbits in its gaps.

The rotation number of $M$ is necessarily rational in Case (i), and necessarily irrational in Case (iii). In Case (ii), $M$ may have either rational or irrational rotation number, as the example of the shear map shows. However, it has been shown (Zehnder (???generic prop of twist maps)) that maps with rational invariant circles are non generic. As for homoclinic and heteroclinic orbits as in (i) and (iii), they have been shown to exist each time there are no invariant circles of the corresponding rotation numbers Hasselblat & Katok (1995) , Mather (1986) .

The feature that is striking in the Aubry-Mather Mather theorem is the possible occurence of Aubry-Mather sets as in (iii). The $F$-invariant Cantor sets have been called *Cantori* by Percival (1979)who constructed them for the discontinuous sawtooth map (a standard map with sawtooth shaped potential). This type of dynamics does occur in twist map, since it can be shown that many maps have no invariant circles, and hence the irrational Aubry-Mather sets must be of type (iii), *i.e.* contain a Cantori.

Although one can construct Aubry-Mather sets that are not made of minimizers, the name "Aubry Mather set" is often reserved to the action minimizing Cantori $M_\omega$ as defined below:

**Proposition 12.9** *For each rotation number $\omega$ there is a unique Cantorus $M_\omega$ made of recurrent minimizing orbits of rotation number $\omega$. The closure of any CO minimizing orbit of rotation number $\omega$ is contained in $M_\omega$.*

*Proof.* A CO minimizing orbit forms an $F$-ordered set, contained in an Aubry-Mather set, and hence conjugated to an orbit of a circle homeomorphism. The closure of the CO minimizing orbit is therefore in a Cantorus, conjugated to the $\omega$–limit set of the circle homeomorphism. As limit of minimizers, this Cantorus is made up of minimizers. We now prove that this Cantorus is unique: suppose not and there are two of them. Exercise 10.5 implies that the (disjoint) union of these two Cantori forms an $F$–ordered set, hence conjugated to a closed invariant set of a circle homeomorphism. Each Cantorus is the $\omega$-limit set of its points. This is a contradiction to the uniqueness of $\omega$ limit sets of circle homeomorphisms proven in Theorem AMthmcircleomlimset. □

## 13.1 Appendix: Cyclically Ordered Sequences and Circle Maps

In this section, we prove Lemma 9.1, and Lemma 9.2. We then recover important facts about circle homeomorphisms and their invariant sets using the language of CO sequences. Part of the proof below is classical, due to Poincaré in his study of circle homeomorphisms.

### A. Proof of Lemmas 9.1 and 9.2

*Proof of Lemma 9.1.* Let $\boldsymbol{x}$ be a CO sequence. We want to prove that the sequence $\{\frac{x_n - x_0}{n}\}_{n\in\mathbb{Z}}$ is a Cauchy sequence as $n \to \pm\infty$. We do the case $n \to +\infty$ here, the case $n \to -\infty$ will follow.

Given $n \in \mathbb{N}$, let $\alpha_n$ be the integer such that:

$$x_0 + \alpha_n \leq x_n \leq x_0 + \alpha_n + 1. \tag{13.1}$$

We prove by induction that

$$x_0 + k\alpha_n \leq x_{kn} \leq x_0 + k\alpha_n + k, \quad \forall k \in \mathbb{N}. \tag{13.2}$$

Indeed, step 1 in the induction is just (13.1) , and if we assume step $k$, *i.e.* (13.2) then, since $\boldsymbol{x}$ is CO, we get

$$x_n + k\alpha_n \leq x_{(k+1)n} \leq x_n + k\alpha_n + k.$$

Using (13.1) this gives $x_0 + (k+1)\alpha_n \leq x_{(k+1)n} \leq x_0 + (k+1)\alpha_n + (k+1)$, which is the step $k+1$ and finishes the induction.

Dividing (13.2) by $k$ we get

$$\alpha_n \leq \frac{x_{kn} - x_0}{k} \leq \alpha_n + 1. \tag{13.3}$$

Since this is true for all $k > 0$,

$$\left|\frac{x_{kn} - x_0}{k} - \frac{x_n - x_0}{1}\right| \leq 1 \Rightarrow \left|\frac{x_{kn} - x_0}{kn} - \frac{x_n - x_0}{n}\right| \leq \frac{1}{|n|}. \tag{13.4}$$

Writing $z_n = \frac{x_n - x_0}{n}$, and assuming $m > 0, n > 0$ we have that

$$|z_n - z_m| \leq |z_n - z_{mn}| + |z_{mn} - z_m| \leq \frac{1}{n} + \frac{1}{m}, \tag{13.5}$$

and hence $z_n, n \in \mathbb{N}$, is a Cauchy sequence whose limit we call $\rho(\boldsymbol{x})$.

To see how the case $n \to -\infty$ follows, let $m \to \infty$ in (13.5) , and multiply by $n$:

$$|x_n - x_0 - n\rho(\boldsymbol{x})| \leq 1. \tag{13.6}$$

Since in all the above we could have replaced $x_0$ by an arbitrary $x_m, m \in \mathbb{Z}$, the following also holds:

$$|x_n - x_m - (n-m)\rho(\boldsymbol{x})| \leq 1 \qquad \forall m, n \in \mathbb{Z}. \tag{13.7}$$

We let the reader check that this last inequality implies that $\lim_{n \to -\infty} z_n = \rho(\boldsymbol{x})$.

The continuity of $\rho$ is also a consequence of Formula (13.6) . Suppose $\boldsymbol{x}^{(j)} \to \boldsymbol{x}$ pointwise as $j \to \infty$. Constructing sequences $\boldsymbol{z}^{(j)}$ as above, and denoting $\rho(\boldsymbol{x}^{(j)}) = \omega_j, \rho(\boldsymbol{x}) = \omega$, (13.6) yields

$$|z_k^{(j)} - \omega_j| \leq \frac{1}{k}, \quad |z_k - \omega| \leq \frac{1}{k}. \tag{13.8}$$

Since $\boldsymbol{z}^{(j)} \to \boldsymbol{z}$, for all $k$ and $\epsilon > 0$,

$$|\omega_j - \omega_i| \leq |\omega_j - z_k^{(j)}| + |z_k^{(j)} - z_k^{(i)}| + |z_k^{(i)} - \omega_i| \leq \frac{2}{k} + \epsilon$$

whenever $i, j$ are big enough. Hence $\{\omega_k\}_{k \in \mathbb{Z}}$ is a Cauchy sequence, whose limit we denote by $\omega$. Letting $j \to \infty$ in (13.8) yields $\omega = \rho(x)$. □

*Proof of Lemma 9.2* Lemma 9.1 implies that $\mathrm{CO}_{[a,b]} \cap \{\boldsymbol{x} \mid x_0 \in [0,1]\}$ is a closed subset of the set:

$$\{\boldsymbol{x} \in \mathbb{R}^{\mathbb{Z}} \mid x_k = x_0 + k\omega + y_k, (x_0, \omega, \boldsymbol{y}) \in [0,1] \times [a,b] \times [-1,1]^{\mathbb{Z}}, \text{with } y_0 = 0\}$$

which is compact for the product topology, by Tychonov's theorem. We let the reader derive a similar proof for $\mathrm{CO}_{[a,b]}/\tau_{1,0}$. □

## B. Dynamics of Circle Homeomorphisms

The orbits of an orientation preserving circle homeomorphism are (by definition!) Cyclically Ordered. From Lemma 9.1, we can deduce the following theorem, due to Poincaré (1985):

**Theorem 13.1** *All the orbits of the lift $F$ of an orientation preserving circle homeomorphism $f$ have the same rotation number, denoted by $\rho(F)$. The rotation number $\rho$ is a continuous function of $F$, where the set of lifts of homeomorphisms of the circle is given the $C^0$ topology.*

*Proof.* We start by a simple but useful lemma.

**Lemma 13.2** *If two* CO *sequences* $\boldsymbol{x}, \boldsymbol{x}'$ *satisfy* $\boldsymbol{x} < \boldsymbol{x}'$ *then* $\rho(\boldsymbol{x}) = \rho(\boldsymbol{x}')$.

*Proof.* The rotation numbers are the respective asymptotic slopes of the Aubry diagram of $\boldsymbol{x}$ and $\boldsymbol{x}'$. If $\rho(\boldsymbol{x}) \neq \rho(\boldsymbol{x}')$, the the Aubry diagram must cross: there must be a $k_0$ and a $k_1$ such that $x_{k_1} > x'_{k_1}$ and $x_{k_1} < x'_{k_1}$. That contradicts $\boldsymbol{x} < \boldsymbol{x}'$. □

Continuing with the proof of Theorem 13.1, since $F$ is increasing, two distinct orbits $\boldsymbol{x}$ and $\boldsymbol{w}$ of $F$ satisfy $\boldsymbol{x} < \boldsymbol{w}$ or $\boldsymbol{w} < \boldsymbol{x}$. From the lemma $\boldsymbol{x}$ and $\boldsymbol{w}$ have same rotation number. If $f_n \to f$ in the $C^0$ topology, then the $f_n$ orbit of a point $x$ (a CO sequence) tends pointwise to the $f$ orbit of $x$. By Lemma 9.1, $\lim \rho(f_n) = \lim \rho(\{f_n^k(x)\}) = \rho(\{f^k(x)\}) = \rho(f)$. □

We now remind the reader about the structure of invariant sets of circle homeomorphisms. Remember that the *Omega limit set* $\omega(x)$ of a point $x$ under a dynamical system $f$ on some space $X$ is the set of limit points of all subsequences $\{x_{k_j}\}$ where $x_k = f^k(x)$ and $k_j \to +\infty$ as $j \to +\infty$, *i.e.* the set of limit points of the forward orbit. Likewise, the *Alpha limit set* $\alpha(x)$ is the set of limit points of the backward orbit. The following theorem, which basically appears in Poincaré (1985), classifies the possible dynamics of circle homeomorphisms:

**Theorem 13.3** *Let $f$ be a circle homeomorphism and $F$ a lift of $f$. If $\rho(F)$ is rational, then, for any $x \in \mathbb{S}^1$, $\omega(x)$ and $\alpha(x)$ are periodic orbits. The orbit of $x$ is either periodic (in which case $x \in \omega(x) = \alpha(x)$) or it is heteroclinic between $\alpha(x)$ and $\omega(x)$.*

*If $\rho(F)$ is irrational, then, for any $x, x' \in \mathbb{S}^1$, $\alpha(x) = \alpha(x') = \omega(x) = \omega(x')$. Call this set $\Omega(f)$. Then $\Omega(f)$ is either the full circle, or a minimal invariant set which is a Cantor set. In the first case any orbit is dense in the circle, and $f$ is conjugated to a rotation by $\rho(F)$. In the second case, either $x \in \Omega(f)$ is recurrent, or it is homoclimic to $\Omega(f)$, a "gap orbit".*

We remind the reader that a *Cantor set* $K$ is a closed, perfect, and totally disconnected topological set. *Perfect* means that each point in $K$ is the limit of some (non constant) sequence in $K$, and *totally disconnected* means that, given any two points $a$ and $b$ in $K$, one can find disjoint closed sets $A$ and $B$ with $a \in A, b \in B$ and $A \cup B = K$. In the real line or the circle, a closed set is totally disconnect if and only if it is nowhere dense. A set $X$ is *nowhere dense* if $Interior(Closure(X)) = \emptyset$.

*Proof of Theorem 13.3*

**Rational rotation number.** Suppose $\rho(F) = m/n$. Then $F^n - m$ must have a fixed point, otherwise for all $x \in \mathbb{R}$, $F^n(x) - x \neq m$ and we can assume $F^n(x) - x > m$. By compactness of $\mathbb{S}^1$, $\rho(F) > m/n$, a contradiction. Hence $F$ has an $m, n$-periodic orbit. By continuity, on any interval $I$ where $F^n - Id - m$ is non zero, it must stay of a constant sign. This sign describes the direction of progress of points inside the orbit of $I$ towards its endpoints: they must be heteroclinic to the endpoint orbits. Conversely, if $F$ has an $m, n$-periodic orbit, its rotation number and thus that of $F$ must be $m/n$.

**Irrational rotation number.** Suppose $\rho(F)$ is irrational. Let $x \in \mathbb{S}^1$ and denote by $\boldsymbol{x} = \{x_k\}_{k\in\mathbb{Z}}$ its orbit under $f$ (with $x = x_0$). Suppose $\omega(x) = \mathbb{S}^1$. Then $\omega(x') = \mathbb{S}^1$ for any other $x' \in \mathbb{S}^1$, otherwise there would be an interval $(a,b)$ not containing any $x'_k = f^k(x')$. But $(a,b)$ would contain some $[x_n, x_m]$ by density of $\boldsymbol{x}$. The intervals $f^{-i(m-n)}[x_n, x_m]$ must cover $\mathbb{S}^1$ and hence $f^{i(m-n)}x' \in (a,b)$ for some $i$, a contradiction. We guide the reader through the proof that $f$ is conjugated to a rotation by $\rho(f)$ in Exercise 13.5.

Suppose $\omega(x) \neq \mathbb{S}^1$. Then, since $\omega(x)$ is closed, its complement is the union of open intervals. Take another point $x'$. We want to show that $\omega(x') = \omega(x)$. We will prove that $\omega(x') \subset \omega(x)$: by symmetry $\omega(x) \subset \omega(x')$. This is obvious if $x' \in \omega(x)$. Suppose not. Then $x'$ is in an open interval $I$ in the complement of $\omega(x)$ whose endpoints are in $\omega(x)$. The orbit of $I$ is made of open intervals in the complement of $\omega(x)$ whose endpoints are orbits in $\omega(x)$. Since there is no periodic orbit, these intervals are disjoint: by the intermediate value theorem $f^k(I) \subset I$ would imply the existence of a fixed point for $f^k$, hence a periodic orbit. The length of these intervals must tend toward 0 under iteration. Thus the orbit of $x'$ approaches the endpoint orbit of $I$ arbitrarily *i.e.* it is asymptotic to $\omega(x)$. Hence $\omega(x') \subset \omega(x)$. In particular $\omega(x) = \Omega(f)$ is a minimal invariant set: any closed invariant subset of $\Omega(f)$ must contain the $\omega$-limit set of any of its point, hence $\Omega(f)$ itself.

We now show that $\Omega(f)$ is a Cantor set. That it is closed is a property of $\omega$-limit sets. It is perfect since $x \in \Omega(f)$ means that $x \in \omega(x)$ and hence $f^{n_k}(x) \to x$ for some $n_k \nearrow \infty$ with all the $f^{n_k}(x)$'s are in $\omega(x)$. To prove that $\Omega(f)$ is nowhere dense, first note that the topological boundary $\partial\Omega(f) = \Omega \backslash Interior(\Omega(f))$ must satisfy $\partial\Omega(f) = \Omega(f)$ or $\partial\Omega(f) = \emptyset$: $\partial\Omega(f)$ is closed, invariant under $f$ and included in $\Omega(f)$ which is a minimal set. But $\partial\Omega(f) = \emptyset$ means $\Omega(f) = Interior(\Omega(f))$ is open, and because it is also closed, it must be all of $\mathbb{S}^1$, which we have ruled out. The alternative is $\partial\Omega(f) = \Omega(f)$, which means $Interior(\Omega(f)) = \emptyset$, what we wanted to prove. □

**Remark 13.4** A circle homeomorphism with an invariant Cantor set cannot be too smooth: Denjoy (see Hasselblat & Katok (1995) , Robinson (1994) ) proved that if $f$ is a $C^1$ diffeomorphism of $\mathbb{S}^1$ with irrational rotation number and derivative of bounded variation, then $f$ has a dense orbit (*i.e.* $\Omega(f) = \mathbb{S}^1$) and is therefore conjugated to a rotation of angle $\rho(F)$. On the other hand, Denjoy did construct a $C^1$ diffeomorphism with $\Omega(f)$ a Cantor set. The idea is simple: take a rotation by irrational angle $\alpha$. Cut the circle at some point $x$ and at all its iterate $f^k(x)$. Glue in at these cuts intervals $I_k$ of length going to 0 as $k \to \infty$, in such a way that the new space you obtain is again a circle. Extend the map $f$ by linear interpolation on the $I_k$. You get a circle homeomorphism with rotation number $\alpha$. With some care, one can make this homeomorphism differentiable, but only up to a point ($C^1$ with Hölder derivative). The complement of the $I_k$'s in the new circle is a Cantor set, which is minimal.

**Exercise 13.5** In this exercise, we prove that if a circle homeomorphism has a dense orbit, then it is conjugated to a rotation.
a) Prove that $\boldsymbol{x}$ is a CO sequence with irrational $\rho(\boldsymbol{x})$ *iff*

$$\forall n, m, p \in \mathbb{Z}, \quad x_n < x_m + p \Longleftrightarrow n\rho(\boldsymbol{x}) < m\rho(\boldsymbol{x}) + p$$

(*Hint.* Use Formula (13.7) for multiples of $m$ and $n$). What is the proper corresponding statement for CO sequences of rational rotation number?
b) Suppose the circle homeomorphism $f$ has a dense orbit $\boldsymbol{x}$. Build a map $h : \mathbb{S}^1 \to \mathbb{S}^1$ by first defining it on $\boldsymbol{x}$ by:

$$x_k \mapsto k\rho(\boldsymbol{x})$$

Use a) to show that $h$ is order preserving and show that its extension by continuity is well defined, has continuous inverse and preserves orbits.

Lemma AMlemmax<yrot is 13.2, Theorem AMtheoremperiodic is 6.3, Section AMsectionlimits is 7, Lemma AMlemmaaubry is 10.2, Corollary AMcorollaryaubry is 10.3, Exercise AMexominordered is 10.5, Lemma AMlemmacoestimate is 9.1, Lemma 11.2is AMthmconvest, Proposition AMpropwmnmin is 10.4, Proposition AMproplipschitz is AMthmlipschitzProposition AMpropmom is 12.9, Theorem AMthmcircleomlimset is 13.3, Formula AMformqgeod is (13.6)