CHAPTER 1 or TWIST

TWIST MAPS OF THE ANNULUS

1/25/2000

Action to be taken: Make and add figures. Correct typos. Revise after writing the intro (some of its material might be used in the intro: standard map, billiard, some definition of symplectic...) The section on elliptic fp may be spread over to Chapter SG. I just moved the section on Poincaré-Birkhofffrom the now defunct Chapter PB. Make sure the transition is smooth. Check out the background, and give reference about Poincaré sections and twist maps.

4. Monotone Twist Maps of the Annulus

A. Definitions

In the first part of this book, we consider diffeomorphisms of the annulus, or of the cylinder. The *annulus* can be defined as

$$\mathbf{A} = \mathbb{S}^1 \times [a, b].$$

[More generally, we could define $\mathbf{A} := \{(x, y) \in \mathbb{S}^1 \times \mathbb{R} \mid u_-(x) \leq y \leq u_+(x)\}$, where both u_- and u_+ smooth functions on $\mathbb{S}^1$]. We define the *cylinder* by:

$$\mathcal{C} = \mathbb{S}^1 \times \mathbb{R}.$$

As with maps of the circle, it is often less ambiguous to work with lifts of diffeomorphisms of $\mathbf{A}$. These are maps of the *strip*:

$$\mathcal{A} := \{(x, y) \in \mathbb{R}^2 \mid a \leq y \leq b\}$$

where x, thought of as the angular variable, ranges over $\mathbb{R}$. The *covering map* $proj : \mathcal{A} \mapsto \mathbf{A}$ takes (x, y) to $(x \bmod 1\ , y)$ and a *lift* of a map f of the annulus is a map F of the strip which satisfies:

$$proj \circ F = f \circ proj.$$

This implies in particular that $F(x+1, y) = F(x, y) + (n, 0)$, for some integer n. By continuity, n does not depend on the point (x, y), nor on the lift F of f, it is called the *degree* of f. In this book, we assume that f is an orientation preserving diffeomorphism of the annulus. In this case, the degree of f is 1 and

$$F(x+1, y) = F(x, y) + (1, 0) \tag{4.1}$$

for any lift F of f. Denoting by T the translation $T(x, y) = (x + 1, y)$, equality (4.1) reads:

$$F \circ T = T \circ F \tag{4.2}$$

Clearly, any map F of $\mathcal{A}$ that satisfies (4.2) is the lift of a map f of $\mathbf{A}$ which has degree 1. We say that f is *induced* by F.

Definition 4.1 Let F be a diffeomorphism of $\mathcal{A} = \mathrm{I\!R} \times [a, b]$ and write $(X(x, y), Y(x, y)) = F(x, y)$. Let F satisfy:

(1) F *preserves the boundaries of* $\mathcal{A}$: $Y(x, a) = a, Y(x, b) = b$.

(2) *Twist Condition:* the function $y \mapsto X(x_0, y)$ is strictly monotone for each given x_0.

(3) *Area and Orientation Preserving:* $\det DF = 1$ or, equivalently, $dY \wedge dX = dy \wedge dx$.

(4) $F \circ T = T \circ F$

Then F induces a map f on the annulus $\mathbf{A}$ which is called a *(area preserving, monotone) twist map of the annulus.*

Exercise 4.1 Prove the above statements about the degree of a map and its lifts.

B. Comments on the Definition

Twist Condition. Condition (2) implies that the map $y \mapsto X(x_0, y)$ is a diffeomorphism between the vertical *fiber* $\{x = x_0\}$ and its image on the x-axis (also called the *base*). In other words, the image of the fiber x_0 by F forms a graph over the x-axis, as is shown in Figure 4. 1.

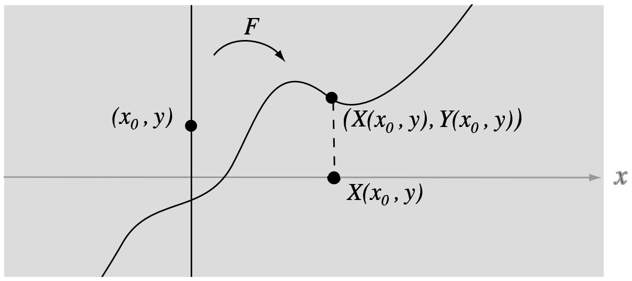

Fig. 4. 1. The positive twist condition: as one moves up along a vertical fiber, the image point moves right.

Often, the monotonicity of the map $y \mapsto X(x_0, y)$ is expressed by the equivalent derivative condition:

$$\frac{\partial X}{\partial y} \neq 0. \tag{4.3}$$

Since $\mathcal{A}$ is connected, this derivative is either always strictly positive, or always strictly negative. We say that F is a *positive twist map* (resp. *negative twist map*) if $y \mapsto X(x_0, y)$ is strictly increasing (resp. decreasing). Note that the lift of a positive twist map "moves" points on the upper boundary of $\mathcal{A}$ "faster" than on the lower boundary. If F satisfies the latter, we say that it has the *boundary twist condition.* This condition, much

weaker than the twist condition in Defi
d in the Poincaré-Birkhoff theorem, see
TWISTsecpb.

We now show that the twist conditi
$, y) \mapsto (x, X)$ is an *embedding* of $\mathcal{A}$ in
$\mathbb{R}^2$, *i.e.* a local diffeomorphism which
rential of ψ is given by :

whose determinant $\frac{\partial X}{\partial y}$ is non zero by
s a local diffeomorphism. To show that
it is injective, suppose $\psi(x_1, y_1) = \psi($
$_2$, and y_1 and y_2 are forced to be equal
because the map $y \mapsto X(x_1, y)$ is stric
he reader to verify that, conversely, if ψ
is an embedding of $\mathcal{A}$, then the twist co
as a change of coordinates.

Area Preservation, Flux and S
of variable formula in multivariate inte-
gration shows that the infinitesimal conc
$ea(X) = Area(F(X))$ for any domain
X in $\mathcal{A}$ (or for any Lebesgue measura
a preservation to another global notion:
that of flux. For F an area preserving m
lefine the function $S : \mathcal{A} \to \mathbb{R}$ by:

where this path integral is over *any* cu
t z_0 and the variable $z = (x, y)$. Using
Stokes' theorem and Condition (3) and
nected, one shows that S is well defined
on $\mathcal{A}$ (*i.e.* it is independent of the path
$- ydx = dS$ (see Exercise 4.2). [This
is just an expression of the fact that clo
nply connected regions]. The *flux* of an
area preserving map F of $\mathbb{R}^2$ satisfyin
:

This makes sense, since, by Stokes'
t (see Exercise 4.3). The flux of the map
F can be seen geometrically in the cylin
d between an embedded circle wrapping
once around $\mathcal{C}$ and its image by the m
e 4. 2). Indeed, $S(\beta(1)) - S(\beta(0)) =$
$\int_\beta YdX - ydx = \int_{F(\beta)} ydx - \int_\beta ydx$
Now take β such that $\beta(1) = T\beta(0)$.

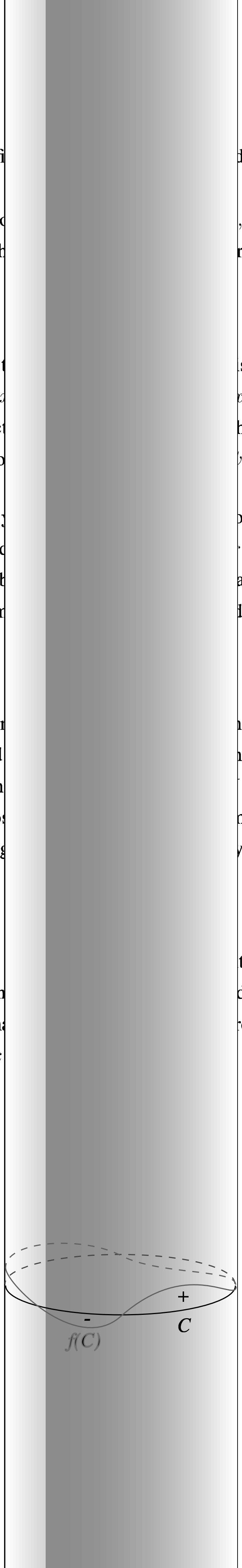

Fig. 4. 2. The flux of a cylinder map as the net area between an enclosing circle C and its image $f(C)$

If F preserves the boundary of a bounded strip $\mathcal{A}$, then f preserves the boundary circles and the flux is by force zero. When no such curve is preserved, the flux can take any value in $\mathbb{R}$ as the example $V_a(x,y) = (x, y+a)$ with $Flux(F) = a$ shows. Since examples of this type show no recurrent dynamics, we exclude them from our study by always imposing, directly or indirectly, the zero flux condition on our maps. If F has zero flux, then $S \circ T = S$ and thus S induces a function s on $\mathbf{A}$ such that

$$f^*ydx - yd = ds. \tag{4.4}$$

taking the exterior derivative on both sides of this equation, one gets $d(f^*ydx - ydx) = d^2s = 0$, and thus

$$f^*(dy \wedge dx) = dy \wedge dx.$$

A map that satisfies this last equality is called *symplectic*, because it preserves the *symplectic form* is called *exact symplectic*. Hence (4.4) shows that exact symplectic implies symplectic. Hence if F has zero flux, the map f it induces is exact symplectic. Conversely, by Stokes' theorem, if f is exact symplectic, any of its lifts has zero flux (Exercise 4.2). Hence the map V_a of the cylinder defined above is not exact symplectic, even though it is symplectic. Note that, in contrast, a symplectic map F of the plane is always exact symplectic: as any closed form on the plane, $F^*(y \wedge dx) - ydx$ is exact (Poincaré's Lemma).

Exercise 4.2 a) Using Stokes Theorem, show that if λ is a closed 1–form on a simply connected domain of $\mathbb{R}^2$, then the function $S = \int_{z_0}^{z} \lambda$ is well defined (*i.e.* does not depend on the path of integration between z and z_0) and that $dS = \lambda$. Apply this to $\lambda = YdX - ydx$.
b) What should a definition of S be if F preserves a smooth area form $\alpha(x,y)dy \wedge dx$?

Exercise 4.3 a) Let F be an area preserving map of $\mathbb{R}^2$ with $F \circ T = T \circ F$. Show that for the function S defined above, $S \circ T - S$ is constant, and hence $Flux(F)$ is well defined. (*Hint.* Given two points z_1, z_2 in $\mathcal{A}$, take any two curves γ_1, γ_2, with γ_i joining z_i and Tz_i, $i = 1, 2$. Take a curve β joining z_1 and z_2 and apply Stokes Theorem to the closed curve $\beta \cdot \gamma_1 \cdot (T\beta)^{-1} \cdot \gamma_2^{-1}$.)
b) Show that any lift of an exact symplectic map of the cylinder has zero flux.
c) (For those who know about DeRham cohomology) Prove that $Flux(F)$ is the result of the pairing of the class $[f^*ydx - ydx]$ in $H^1_{DR}(\mathcal{C})$ with the first homology class represented by a circle going around the cylinder once in the positive direction (as usual, f is the map induced by F).

C. Twist Maps of the Cylinder

The comments of the previous subsection motivate the following:

Definition 4.4 (Twist Maps of the Cylinder) Let F be a diffeomorphism of $\mathbb{R}^2$ and write $(X(x,y), Y(x,y)) = F(x,y)$. Let F satisfy:

(1) *F is isotopic to the Identity*
(2) *Twist Condition:* the map $\psi := (x,y) \mapsto (x, X(x,y))$ is a diffeomorphism of $\mathbb{R}^2$
(3) *Area Preserving & Zero Flux (Exact Symplectic):* $YdX - ydx = dS$ with some real valued function S on $\mathbb{R}^2$ satisfying:

$$S(x+1,y)=S(x,y).$$

Then F is the lift of a map f on the cylinder $\mathcal{C}$ which is called a *monotone twist map of the cylinder.*

Condition (1) means that F can be deformed continuously into the identity through a path of homeomorphisms of the cylinder. For maps of the closed strip $\mathrm{I\!R}\times[a,b]$, this condition clearly implies that the boundaries have to be preserved, and hence Condition (1) here is the analog to Condition (19) in Definition 4.1. It will appear clearly in next section that the periodicity of the function S implies the periodicity $F\circ T=T\circ F$, $i.e.$ Condition (4) of Definition 4.1, which is necessary for F to induce a map of the cylinder. Finally, the condition that ψ be a diffeomorphism here can be relaxed: one can require that ψ only be an embedding, $i.e.$ a diffeomorphism of $\mathrm{I\!R}^2$ into a proper subset of $\mathrm{I\!R}^2$, to the cost of some (manageable) complications.

Remark 4.5 There exist several other definitions of monotone twist maps in the literature. Most noteworthy are the topological definitions, where the map is only required to be a homeomorphism (and not necessarily a diffeomorphism). The twist condition takes different forms with different authors. One commonly used is that the map $y\mapsto X(x,y)$ be monotonic (Boyland (1988), Hall (1984), Katok (1982), LeCalvez (astérisque)). A much milder condition is considered in Frank (1988), where certain neighborhoods must move in opposite directions around the annulus. The preservation of area is sometimes discarded by these authors, replaced by a condition that the map contracts the area, or that it is topologically recurrent. The topological theory for twist maps is extremely rich and would be the subject of an entire book. Our choice of working in the differentiable category stems from the possibilities of generalization to higher dimensions that it offers.

Exercise 4.3 Show that a map of the bounded annulus which is homotopic to Id preserves each boundary component (Note: the converse is also true, but much harder to prove).

5. Generating Functions and the Variational Setting

A. Generating Functions

In the previous section, we have seen that the lift F of a twist map of either the cylinder or the annulus comes with a function S such that $F^*ydx-ydx=YdX-ydx=dS$ and $S(x+1,y)=S(x,y)$. The first equation expresses the fact that F preserves the area, whereas the periodicity of S, expresses the zero flux condition.

On the other hand, the twist condition on F gives us a function ψ which we view as a change of coordinates $\psi:(x,y)\mapsto(x,X)$. In the (x,X) coordinates$^{(2)}$ the equation $YdX-ydx=dS(x,X)$ implies immediately that the functions $-y(x,X)$ and $Y(x,X)$ are the partial derivatives of S:

$$y=-\frac{\partial S(x,X)}{\partial x},\qquad Y=\frac{\partial S(x,X)}{\partial X} \tag{5.1}$$

These simple equations are the cornerstone of this book. The function $S(x,X)$ is called the *generating function* of F in that from S we can retrieve F, at least implicitly: ψ^{-1} is given by $(x,X)\mapsto(x,-\frac{\partial S}{\partial x})$ hence ψ is implicitly given by S. Thus F is defined by:

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[2] Remember that under the change of coordinates ψ, a function S changes according to $S\mapsto S\circ\psi$. Likewise, $y\mapsto y\circ\psi$ and $Y\mapsto Y\circ\psi$.

$$F:(x,y)\mapsto(X\circ\psi(x,y),\frac{\partial S}{\partial X}(\psi(x,y))) \tag{5.2}$$

and the two coordinates of F are given implicitly by the function S and its partial derivatives. In Proposition PROPgfstm of Chapter STM, we give conditions under which a function on $\mathbb{R}^2$ is a generating function of the lift F of some twist map. We also show that the correspondence between maps and their generating functions (mod constant) is one to one and continuous. The following exercise gives two necessary conditions for a function to generate a twist map:

Exercise 5.1 Show that if $S(x,X)$ is the generating function of a positive twist map, then:
a) $\partial_{12}S(x,X)<0$
b) $S(x+1,X+1)=S(x,X)$

Exercise 5.2 Show that if F the lift of a twist map of the annulus $\mathbb{S}^1\times[0,1]$ then $S(x,X)$ can be interpreted as the area of the triangular shaped area with vertices $(x,0)$, $(X,0)$ and (X,Y) shown in Figure 5. 1. (*Hint.* Show geometrically on this picture that $Y=\frac{\partial S}{\partial X}$. For $y=-\frac{\partial S}{\partial x}$, consider the preimage of this triangular region by F). Solve question b) of the previous exercise using this geometric construction.

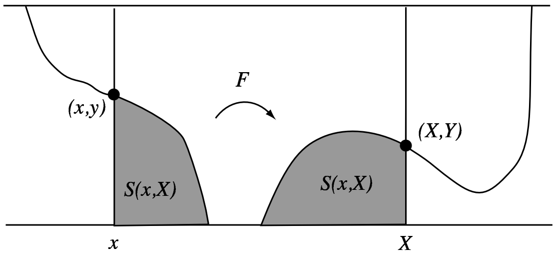

Fig. 5. 1. The generating function as an area

Exercise 5.3 Show that the inverse of a positive twist map with generating function $S(x,X)$ is a negative twist map with generating function $-S(X,x)$.

B. Variational Principle

The lift F of a twist map gives rise to a dynamical system whose orbits are given by the images of points of $\mathbb{R}^2$ under the successive iterates of F. The *orbit* of the point (x_0,y_0) is the biinfinite sequence:

$$\{\dots(x_{-1},y_{-1}),(x_0,y_0),(x_1,y_1),\dots,(x_k,y_k)\dots\}$$

where $(x_k,y_k)=f(x_{k-1},y_{k-1})$.

Lemma 5.1 *Let F be a monotone twist map of $\mathcal{A}$ or $\mathbb{R}^2$ and let $S(x,X)$ be its generating function. There is a one to one correspondence between orbits $\{(x_k,y_k)=f^k(x_0,y_0)\}_{k\in\mathbb{Z}}$ of F and sequences $\{x_k\}_{k\in\mathbb{Z}}$ satisfying:*

$$\partial_1S(x_k,x_{k+1})+\partial_2S(x_{k-1},x_k)=0 \qquad \forall k\in\mathbb{Z}. \tag{5.3}$$

The correspondence is given by: $y_k = -\partial_1 S(x_k, x_{k+1})$.

Proof. Let $\{(x_k, y_k)\}_{k\in\mathbb{Z}}$ be an orbit of F. Since $(x_k, y_k) = f(x_{k-1}, y_{k-1})$ for all integer k, Equation (5.1) implies:

$$y_k = -\partial_1 S(x_k, x_{k+1}) = \partial_2 S(x_{k-1}, x_k).$$

Conversely, let $\{x_k\}_{k\in\mathbb{Z}}$ satisfy Equation (5.3) and set $y_k = -\partial_1 S(x_k, x_{k+1})$, for all integer k. Then, applying Equations (5.2) and (5.3) :

$$\begin{aligned} f(x_{k-1}, y_{k-1}) &= f \circ \psi^{-1}(x_{k-1}, x_k) = (x_k, \partial_2 S(x_{k-1}, x_k)) \\ &= (x_k, -\partial_1 S(x_k, x_{k+1})) = (x_k, y_k). \end{aligned}$$

□

Equations (5.3) can be interpreted as "discrete Euler-Lagrange" equations for some action function on the space of sequences. Indeed, let F be the lift of a twist map of the cylinder, and $S(x, X)$ its generating function. Given a sequence of points $\{x_N, \dots, x_M\}$, we can associate its *action* defined by:

$$W(x_N, \dots, x_M) = \sum_{k=N}^{M-1} S(x_k, x_{k+1})$$

Corollary 5.2 (Critical Action Principle) *A sequence* $\{x_N, \dots, x_M\}$ *is the projection of an orbit segment of* F *on the* x*-axis if and only if it is a critical point of* W *restricted to the subspace of sequences* $\{w_N, \dots, w_M\}$ *with fixed endpoints:* $w_N = x_N, w_M = x_M$.

Proof. Given a sequence $\{x_N, \dots, x_M\}$, introduce the sequences

$$y_k = -\partial_1 S(x_k, x_{k+1}) \quad \text{and} \quad Y_k = \partial_2 S(x_k, x_{k-1}).$$

In particular, $F(x_k, y_k) = (x_{k+1}, Y_k)$. If $\tilde{W}$ is the restriction of W to the set of sequences with fixed endpoints x_N and x_M, a direct calculation yields:

$$d\tilde{W}(x_N, \dots, x_M) = \sum_{k=N+1}^{M-1} (Y_{k-1} - y_k) dx_k.$$

Hence $\{x_N, \dots, x_M\}$ is a critical point for W if and only if $Y_{k-1} = y_k$, which is a rephrasing of Equation (5.3) , *i.e.* the sequence $\{(x_N, y_N), \dots, (x_M, y_M)\}$ is an orbit segment. □

Exercise 5.4 Adapt Lemma 5.1 to a situation where the map F is a composition of different twist maps $F = F_k \circ \dots \circ F_1$ with generating functions $S_1, \dots, S_k$. Note that you do not need to assume that all the F_i are either positive twist (or all negative twist). If they are, one calls F a positive (*resp.* negative) *tilt map*.

C. Periodic Orbits

Let F be the lift of a twist map f of the annulus $\mathbf{A}$, or cylinder $\mathcal{C}$. Suppose that some orbit $\{x_k, y_k\}_{k\in\mathbb{Z}}$ of F satisfies:

$$x_{k+n} = x_k + m \tag{5.4}$$

that is, $F^n(x_k, y_k) = T^m(x_k, y_k)$. Then $f^n(proj(x_k, y_k)) = proj(x_k, y_k)$, and thus the orbit of (x_0, y_0) is the lift of a periodic orbit of f. We say that a sequence $\{x_k\}$ satisfying (5.4) is a (m, n) *sequence*. An orbit whose x projection is an (m, n) sequence is called a (m, n) *orbit*, or an orbit of *type* (m, n). Hence, under n iterates of F, points in a (m, n) orbit get translated by the integer m in the x direction. Down in the annulus, this can be interpreted as the orbit wrapping m times around the annulus in n iterates. Conversely, it is not hard to see that any periodic orbit of f of period n lifts to an (m,n) orbit of a lift F, for some integer m which *does* depend on the choice of F. The proof of the following is identical to that of Corollary 5.2:

Proposition 5.3 *A (m, n) periodic sequence is the x- projection of a m, n periodic orbit if and only if its is a critical point of $W(x_k, \dots, x_{k+q}) = \sum_{j=k}^{k+q-1} S(x_j, x_{j+1})$ for one (and hence for all) $k \in \mathbb{Z}$.*

Exercise 5.5 Show by an example that the number m for a periodic orbit of a twist map depends on the lift.

D. Rotation Numbers

Another interpretation of the numbers m, n in a periodic orbit is that the average displacement in the x direction of the points in a (m, n) orbit is m/n. In general, if $\{x_k, y_k\}_{k\in\mathbb{Z}}$ is *any* orbit, one can try to compute the limits:

$$\lim_{k\to+\infty} \frac{x_k}{k}, \qquad \lim_{k\to-\infty} \frac{x_k}{k}$$

If these limits exist, they are called respectively the *forward* and *backward rotation numbers*. If they are equal, they are called the *rotation number*. Since $\lim_{k\to\infty} \frac{x_k}{k} = lim_{k\to\infty} \frac{x_k - x_0}{k}$, the rotation number is an asymptotic measure of the average displacement in the x direction along an orbit. Obviously, an (m, n) periodic orbit has rotation number m/n. We also call *rotation number of the point* $z = (x, y)$ the rotation number of its orbit under F; we denote this number by $\rho_f(z)$.

Exercise 5.6 For those who know Birkhoff's ergodic theorem, show that, if f is an area preserving map of the annulus, $\rho_f(z)$ exists for a set of points z of full Lebesgue measure in $\mathcal{A}$ (*Hint.* $\lim \frac{x_k - x_0}{k} = \lim \frac{1}{k} \sum_1^k (x_j - x_{j-1})$ is the time average of some function).

6. Examples

A. The Standard Map

As noted in the introduction, one of the most widely studied family of monotone twist maps is the so called standard family, or *standard map*. We show how to retrieve explicitly the standard map from its generating function. Let

$$S(x,X)=\frac{1}{2}(X-x)^2+V(x),$$

where V is 1-periodic in x. Define

$$\begin{aligned} y &= -\partial_1 S(x,X)=X-x+V'(x)\\ Y &= \partial_2 S(x,X)=X-x. \end{aligned}$$

then it is easily seen that

$$\begin{aligned} X &= x+Y\\ Y &= y+V'(x), \end{aligned}$$

That is, S generates the lift of a twist map:

$$F(x,y)=(X,Y)=(x+y+V'(x),y+V'(x)).$$

Taking as "potential" V the 1-parameter family $\frac{k}{4\pi^2}cos(2\pi x)$, we do indeed get the standard family:

$$F_k(x,y)=(x+y-\frac{k}{2\pi}sin(2\pi x),y-\frac{k}{2\pi}sin(2\pi x))$$

When $V\equiv 0$ (or k is equal to 0 in the standard family), the generating function is $\frac{1}{2}(X-x)^2=\frac{1}{2}\mathrm{Dis}^2(x,X)$ and the map it generates is the *shear map*:

$$F_0(x,y)=(x+y,y)$$

which is *completely integrable*, in the sense that each horizontal line $\{y=y_0\}$ (covering a circle in $\mathcal{C}$) is invariant under F_0, and that the restriction of F_0 to $\{y=y_0\}$ is a translation: $x\mapsto x+y_0$ (lift of a rotation of angle $2\pi y_0$). We will see in Chapter HAM that F_0 is the time 1 map of the geodesic flow for the Euclidean metric on the circle.

As noted in the introduction, an important question about the standard family (or any set of maps containing a completely integrable one) is: which features of F_0 survive as one perturbs the parameter k away from 0?

Exercise 6.1 Check all the axioms of twist maps of the cylinder on the standard map.

B. Elliptic Fixed Points of Area Preserving Maps

The study of the dynamics around conservative elliptic fixed points was the motivation behind the birth of twist maps. It started when Poincaré studied the dynamics around an elliptic periodic orbit in the restricted 3–body problem. This is a Hamiltonian system (see Chapter SG) with 2 degrees of freedom, whose energy surface is 3–dimensional. Poincaré considered the return map on a 2–dimensional transverse section to the periodic orbit. Since the system is Hamiltonian, the return map is symplectic (see Theorem THMhamsym of Chapter SG). Generically, it is also shown to satisfy a twist condition. To formalize this a little, we present here the Birkhoff Normal Form Theorem. Poincaré was interested in proving that an elliptic periodic orbit is stable (leading to the more difficult question of the stability of the solar system), and in finding many periodic orbits close by. Both these problems were solved affirmatively for generic maps, the first by the KAM theory (see INVchapter) and the second by the theorem of Poincaré-Birkhoff (see TWISTsecpb).

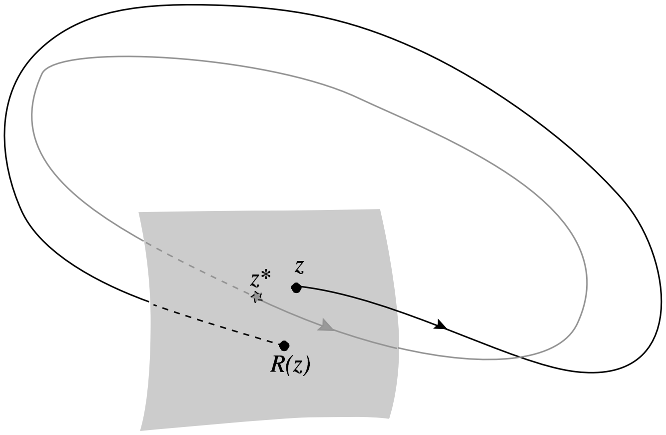

Fig. 6. 2. A Poincaré section around the periodic orbit of the point z^*, with the return map R.

Let F be a symplectic C^∞ diffeomorphism in a neighborhood of 0 in $\mathbb{R}^2$, which has 0 as a fixed point. Since $\det\, Df(0) = 1$, the two eigenvalues are either real $\lambda, 1/\lambda$ or complex $\lambda, \overline{\lambda}$ and conjugated on the unit circle. In the first case, we say that 0 is a *hyperbolic fixed point*, in the second case that it is an *elliptic fixed point* (see also Appendix 1 or SG). If F is the return map of a periodic orbit based at z^* as above, the periodic orbit is called elliptic or (resp. hyperbolic) when z^* is an elliptic (resp. hyperbolic) fixed point for R.

Suppose now that 0 is an elliptic fixed point and that $Df(0)$ has eigenvalues $\lambda = e^{i2\pi\alpha}$ and $\overline{\lambda}$ (i.e. Df(0) is a rotation of angle α). Suppose moreover that $\lambda^n \neq 1$ for n in $\{1, \dots, q\}$ for some integer q. We can make a change of variable $z = x + iy, \overline{z} = x - iy$ and write the Taylor expansion of order n of $F(z)$ in these coordinates:

$$f(z) = \sum_{k=1}^{n} R_k(z, \overline{z}) + o(|z|^n)$$

Theorem 6.1 (Birkhoff Normal Form) *There exists a symplectic (for the form $dx \wedge dy$), C^∞ diffeomorphism h, defined near 0 and having 0 as a fixed point such that:*

$$h \circ f \circ h^{-1}(z) = \lambda z e^{i2\pi P(z\overline{z})} + o(|z|^{q-1})$$

or, in polar coordinates ($z = re^{i2\pi\theta}$):

$$\tilde{f} = \overline{h} \circ f \circ \overline{h}^{-1}(r, \theta) = \big(\theta + \alpha + P(r^2) + o(|r|^{2n}),\ \ r + o(|r|^{2n})\big)$$

where $P(x) = a_1 x + \ldots + a_m x^m$ with $2m+1 < q$. Each of the "Birkhoff invariants" a_k is generically non zero.

For a proof of this, we refer to LeCalvez (1990) . There are also versions that require less differentiability (see Moser (1973)). The point of this theorem is that, if we make the generic assumption that some a_k is non zero, the map F satisfies a twist condition in a neighborhood of $r = 0$ (for $r > 0$). Note that, in polar coordinates, the map $\tilde{f}$ preserves the form $rd\theta \wedge dr$, (which is only non–degenerate for $r > 0$. By making a further change of variables that preserves the vertical foliation $\{x = ct\}$, one can get a map that preserves $d\theta \wedge dr$ (see Chenciner (1985)). This last map preserves no boundaries. However, one can extend it to a boundary preserving map of a compact annulus. The main results in the theory can often be made precise enough to tell apart the dynamics of the original map from that of the extension. Hence the dynamical study around conservative fixed points reduces to the study of twist maps.

C. The Frenkel-Kontorova Model

The variational approach in Section 5 was encountered by Aubry (see Aubry & Le Daeron (1983)) while studying a model in condensed matter physics. In this model, one considers a chain of particles whose nearest neighbor interaction is represented by springs. The chain of particles lies on the surface of a linear crystal represented by a periodic potential $V(x) = k/4\pi^2 cos(2\pi x)$.

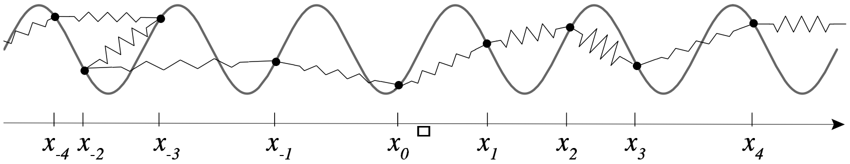

Fig. 6. 4. The Frenkel–Kontorova Model.

If x_k represents the location of the kth particle of the chain, this particle is in equilibrium whenever the sum of the forces applied to it is null:

$$(x_{k+1} - x_k) - (x_k - x_{k-1}) - \frac{k}{2\pi} sin(2\pi x_k) = 0 \tag{6.1}$$

This equation can be rewritten $dW = 0$ where W, the energy of the configuration of particles is given by :

$$W = \sum_k S(x_k, x_{k+1}) = \sum_k \frac{1}{2}(x_k - x_{k+1})^2 + \frac{k}{4\pi^2} cos(2\pi x_k).$$

We recognize S as the generating function of the Standard map. Hence *equilibrium states of the Frenkel-Kontorova model are in 1-1 correspondence with orbits of the Standard map.*

D. Billiard Maps

We revisit here the example of the billiard map presented in the introduction. Consider the dynamics of a ball in a convex, planar billiard. This ball is subject to simple laws : it goes in straight lines between two rebounds and the incidence and reflexion angles are equal at a rebound. We reproduce here a figure of the introduction:

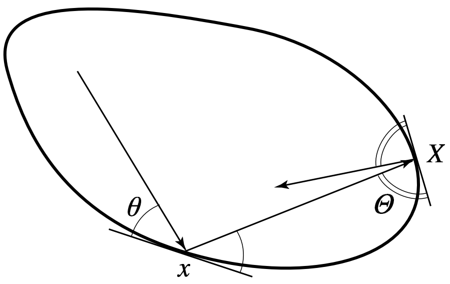

Fig. 6. 5. In a convex billiard, the point x and angle θ at a rebound uniquely and continuously determines the next point X and incidence angle Θ.

Let x be the arc length coordinate with respect to a given point on the boundary C of the billiard, which we orient counterclockwise. Let $y = -cos(\theta)$ where θ is the reflexion angle of a point of rebound. Because of the convexity of the billiard and the law of reflexion, a pair (x, y) at a rebound determines its successor (X, Y), and vice versa. Hence we have constructed a homeomorphism $f : (x, y) \mapsto (X, Y)$ of the (open) annulus $\mathbb{S}^1 \times (-1, 1)$ which is actually a C^{k-1} diffeomorphism if the boundary is C^k (LeCalvez (1990)). We call f the *billiard map.* If we increase y while keeping x fixed, the convexity of C implies that $C(X)$ moves in the positive direction along C. Thus:

$$\frac{\partial X}{\partial y} > 0 \tag{6.2}$$

and the billiard map satisfies the positive twist condition.

We now show that f is exact symplectic by exhibiting a generating function for it. Let $S(x, X) = \|C(X) - C(x)\|$ then, since $C' = \frac{dC}{dx}$ is a unit tangent vector:

$$\begin{aligned}\frac{\partial S}{\partial x} &= \frac{1}{S(x,X)}\left[C'(x).(C(x) - C(X))\right] = y\\ \frac{\partial S}{\partial X} &= \frac{-1}{S(x,X)}\left[C'(X).(C(X) - C(x))\right] = -Y\end{aligned} \tag{6.3}$$

which is to say:

$$YdX - ydx = -dS(x, X) \tag{6.4}$$

Thus, for the billiard map, the action function $W = \sum S(x_k, x_{k+1})$ is nothing more than the perimeter of the trajectory segment considered. For instance, periodic trajectories correspond to polygons in a given p, q-type who are critical points for the perimeter function. Figure 6. 6 shows that a period 5 orbit might come in different orders.

Exercise 6.2 Show that the billiard map for the round billiard is given by:

$$f(x, y) = (x + 2cos^{-1}(-y), y).$$

Exercise 6.3 Show that, for the billiard map, the equation $dW = 0$ expresses the equality between the angle of incidence and the angle of reflexion at each rebound.

7. The Poincaré-Birkhoff Theorem

In this section, we give a complete proof of the Poincaré-Birkhoff theorem, also called Poincaré's last theorem. We refer to Section 3.0for some motivation for this theorem. We use here some elementary material about circle diffeomorphisms, which the reader can get familiarized with in the appendix at the end of Chapter AM. We also use techniques of Conley for the gradient flow of the action function that the reader can read about in Appendix 2 or TOPO. We consider a map f of the compact annulus $\mathbf{A} = \mathbb{S}^1 \times [0, 1]$ and its lift F to $\mathcal{A} = \mathbb{R} \times [0, 1]$. We do *not* assume that f is a twist map, but rather that the restriction of F to each boundary component $u_\pm$, which are lifts of circle diffeomorphisms, have rotation numbers $\rho_\pm$ of $F\big|_{u_\pm}$ which satisfy $\rho_- < \rho_+$ (See . We say that F satisfies the *boundary twist condition.*

Theorem 7.1 (Poincaré-Birkhoff) *The lift F of an area preserving map of $\mathcal{A}$ which satisfies the boundary twist condition with $\rho_- < 0 < \rho_+$ has at least two fixed points. More generally, if $m/n \in [\rho_-, \rho_+]$, and m, n are coprime then F has at least two m, n–orbits.*

Proof. We follow the proof of LeCalvez (astérisque) , which is based on the following simple lemma:

Lemma (Decomposition) 7.2 *Any area preserving map f of a bounded annulus $\mathbf{A}$ isotopic to the Identity, can be written as a composition of twist maps:*

$$f = f_{2K} \circ \ldots \circ f_1$$

Proof. It is a general fact (left as an exercise to the reader) about topological groups that the connected component of the neutral element is generated by finite products of elements in any given neighborhood U of the neutral element of the group. Let f_0 be the shear map $f(x, y) = (x + y \mod 1, y)$. Since the set of maps satisfying the twist condition is open, there is a neighborhood U of Id in the set of area preserving maps of $\mathbf{A}$ which is such that $f \in U \Rightarrow f_0^{-1} \circ f$ is a negative twist map . Hence any f in U can be written as: $f = f_0 \circ (f_0^{-1} \circ f)$, a composition of two twist maps (one positive, the other negative). The group of area and orientation preserving maps of the annulus being connected, any map in that group can be written as a finite combinations of f as above. □

Let f be area preserving and let F be a lift of f to the covering space $\mathcal{A}$. Then $F = F_{2K} \circ \ldots \circ F_1$ where F_k lifts a twist map f_k. Let S_k be the generating function for F_k. If we let

$$W_0(\boldsymbol{x}) = \sum_{k=1}^{2K} S_k(x_k, x_{k+1}) \qquad \boldsymbol{x} \in X_{0,2K} = \{x_{2K+l} = x_l\}$$

then the Critical Action Lemma 5.3 shows that the critical points of W_0 correspond to periodic orbits under the successive f_k's, and hence to fixed points of f. To find these critical points, we study the gradient flow ζ^t of $-W_0$ and exhibit a compact set P of $X_{0,2K}$ which must contain critical points for the action. The set P is an *isolating block* in the sense of Conley, *i.e.* a compact neighborhood whose boundary points immediately exit P in (small) positive or negative time (see Appendix 2 or TOPO). This condition on the boundary implies that the maximum invariant set for ζ^t is in the interior of P (hence the term "isolating").

Lemma 7.3 *Whenever $\rho_- < 0 < \rho_+$, the set*

$$P = \{\boldsymbol{x} \in X_{0,2K} \mid 0 \leq -\partial_1 S_k(x_k, x_{k+1}) \leq 1\}$$

is an isolating block for the gradient flow ζ^t of $-W_0$. Moreover,

$$P \simeq \mathbb{S}^1 \times [0,1]^K \times [0,1]^{K-1}$$

with exit set $P^- = \mathbb{S}^1 \times [0,1]^K \times \partial([0,1]^{K-1})$

Proof. Setting $y_k = -\partial_1 S_k(x_k, x_{k+1})$, the faces of the boundary ∂P of P can be written as $\{y_k = 0\}$ or $\{y_k = 1\}$ for $k \in \{1, \ldots, 2K\}$. The behavior of the flow at a face $y_k = 1$, say, is given by the sign of $\frac{dy_k}{dt} = \dot{y}_k$:

$$\dot{y}_k = -\frac{d}{dt}\left(\partial_1 S_k(x_k, x_{k+1})\right) = -\partial_{11} S_k(x_k, x_{k+1})\dot{x}_k - \partial_{12} S_k(x_k, x_{k+1})\dot{x}_{k+1} \tag{7.1}$$

We let $Y_k = \partial_2 S_k(x_k, x_{k+1})$, *i.e.* $F_k(x_k, y_k) = (x_{k+1}, Y_k)$. With this notation $-\frac{\partial W_0}{\partial x_k} = Y_{k-1} + y_k$, and Equation (7.0) reads:

$$\dot{y}_k = \partial_{11} S_k(x_k, x_{k+1})(Y_{k-1} - y_k) + \partial_{12} S_k(x_k, x_{k+1})(Y_k - y_{k+1}) \tag{7.2}$$

and the invariance of the boundary component $\mathbb{R} \times \{1\}$ of $\mathbb{R} \times [0,1]$ under F_k tells us that, when $y_k = 1$ then $Y_k = 1$ as well. Since $y_{k\pm1} \leq 1$ and hence $Y_{k-1} \leq 1$,

$$Y_{k-1} - y_k \leq 0, \quad Y_k - y_{k+1} \geq 0. \tag{7.3}$$

Assume that k is even. Then f_k is a positive twist map and $-\partial_{12} S_k(x_k, x_{k+1}) > 0$. We need to determine the sign of $\partial_{11} S(x_k, x_{k+1})$ on the subset $\{y_k = 1\}$ of ∂P. On this set, we have $x_k = a(x_{k+1})$ where a is the restriction of F_k^{-1} to $y = 1$, this latter set being parameterized by x. Since a is the lift of an orientation preserving circle diffeomorphism, we have $a'(x) > 0$ for all x. We differentiate the equation $1 = \partial S(a(x), x)$ with respect to x :

$$0 = a'(x)\partial_{11} S(a(x), x) + \partial_{12} S(a(x), x)$$

from which we deduce that $\partial_{11} S(x, a(x)) > 0$. Going back to Equation (7.2) , we see that if we are away from the boundary of the face $y_k = 1$ (i.e., in particular, $y_l \neq 1,\ l = k-1, k+1$), then the inequalities in (7.3) are strict, and we get $\dot{y}_k < 0$: the flow is strictly entering P through this face, or exiting it in negative time.

If we are on an edge of the face $y_k = 1$, the inequalities (7.3) may be equalities. But this cannot be the case for all k: if it were, $(x_k)_{k\in\mathbb{Z}}$ would be critical and (x_k, y_k) would be a fixed point for f on the boundary, which is impossible since then the rotation number $\rho_+ = 0$, a contradiction to $\rho_- < 0 < \rho_+$. So we can assume, say $Y_{l-1} - y_l < 0$, $y_l = y_{l+1} = \ldots = y_k = 1$, in which case (7.2) tells us that $\dot{y}_l \neq 0$ and the flow exits P in either positive or negative time at this point of ∂P.

The proof of the case k odd is exactly similar. We let the reader show in Exercise 7.4 that P and its exit set P^- have the topology advertised. □

This Lemma puts us in a situation which, since the work of Conley & Zehnder (1983) is a classic one in the field of symplectic topology. It can be schematized by the following diagram:

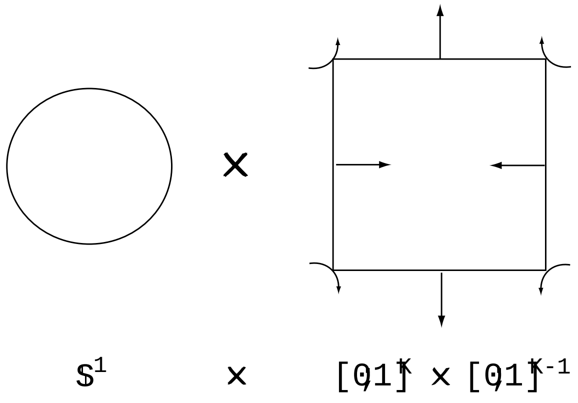

Fig. 7. 2. The gradient flow at the boundary of the isolating block P

Given this topological data on its gradient flow, Proposition 50.3 tells us that W_0 must have at least $cl(\mathbb{S}^1) = 2$ of critical points. This completes the proof of the Poincaré-Birkhoff Theorem. The more general case of periodic orbits with rotation number $m/n \in (\rho_-, \rho_+)$ derives from the fixed point case by considering the map $F^n(\cdot) - (m, 0)$, which has new rotation numbers on the boundary $n(\rho_- - m/n) < 0 < n(\rho_+ - m/n)$ and whose fixed points correspond to m, n periodic orbits of F. □

Corollary CORvarprin or TWISTcorvarprin is 5.2, Proposition TWISTpropcritperiod is 5.3, Section TWISTsectionvariation is 5, Section TWISTsecexamples is 23,Section TWISTsecpb is 26