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# SYMPLECTIC TWIST MAPS

## Global Variational Techniques

To Jürgen Moser,
A kind man of great dignity, whose influence on this field,
and on Mathematics in general, was greater than any list of
publications will ever account for.

# FOREWORD

Area preserving maps of the annulus first appeared in the work of Henri Poincaré (1899) (see also Poincaré (1912)) on the three-body problem. As two dimensional discrete dynamical models, they offered a handle for the study of a complicated Hamiltonian system. Since then, these maps and their more specialized offspring called twist maps, have offered many opportunities for the rigorous analysis of aspects of Hamiltonian systems, as well as an ideal test ground for important theories in that field (*eg.* Moser (1962) proved the first differentiable version of the KAM theorem in the context of twist maps).

This book is intended for graduate students and researchers in mathematics and mathematical physics interested in the interplay between the theories of twist maps and Hamiltonian dynamical systems. The original mandate of this book was to be an edited version of the author's thesis on periodic orbits of symplectic twist maps of $\mathbb{T}^n \times \mathbb{R}^n$ . While it now comprises substantially more than that, the results presented, especially in the higher dimensional case, are still very much centered around the author's work.

At the turn of the 1980's, the theory of twist maps received a tremendous boost from the work of Aubry and Mather. Aubry, a solid-state physicist, had been led to twist maps in his work on ground states for the Frenkel-Kontorova model. This system, which models deposition on periodic 1–dimensional crystals, while not dynamical, provides a variational approach which is surprisingly relevant to twist maps. Mather, a mathematician who had worked on dynamical systems and singularity theory, gave a proof of existence of orbits of all rotation numbers in twist maps, what is now known as the Aubry-Mather theorem, using a different variational approach proposed by Percival. Aubry, who had conjectured the result, gave a proof using his approach. [It is interesting to note that Hedlund (1932) had developped long before a very similar theory in the context of minimal geodesics on the torus. Bangert (1988) unified the two theories.] Both researchers then developed a sophisticated body of work using an interplay of their two approaches. This lead to a flurry of work in mathematics and physics.

At about the same time, Conley & Zehnder (1983) gave a proof of the Arnold conjecture on the the torus, which heralded the birth of symplectic topology. This conjecture (now a theorem) states that the number of fixed point for a Hamiltonian map on a closed symplectic manifold is closely related to the minimum number of critical points of real valued functions on that manifold. The proof involved Conley's generalized Morse theory for the study of the gradient flow of the Hamiltonian action functional in loop space. Later, with the influx of Gromov's holomorphic curve theory, this gave rise to Floer cohomology (Floer (1989b) ). Interestingly, Arnold (1978) introduced his conjecture as a generalization of the famous fixed point theorem for annulus maps of Poincaré and Birkhoff, by gluing two annuli into a torus.

This book aims at relating these two historical currents: while establishing a firm ground in the classical theory of twist maps, the text reaches out, via generalized symplectic twist maps, to Hamiltonian systems and symplectic topology. One of the approaches used throughout is that of the gradient flow of the action functional stemming from the twist maps' generating functions. We hope to convey that symplectic twist maps offers a relatively simple, often finite dimensional, interface to the variational and dynamical study of Hamiltonian systems on cotangent bundles.

Results for the two dimensional theory presented here include the classical theorems by Poincaré, Birkhoff (Chapter 1 and Chapter 6), Aubry and Mather (Chapter 2). A joint work of the author with Sigurd Angenent on the vertical ordering of Aubry-Mather sets appears for the first time here (Chapter 3). The approach of this book to the two dimensional theory is deliberately variational (except for Katznelson and Ornstein recent proof of Birkhoff's Graph Theorem in Chapter 6) as I sought continuity between the low and high dimensions. Unfortunately, this choice leaves out the rich topological theory of twist maps and, more generally two dimensional topological dynamics. I refer the reader interested in the topological approach to Hall & Meyer (1991), LeCalvez (1990) and the bibliography therein.

In higher dimensions, results by the author form the main focus of attention. These results are about the existence of periodic orbits and their multiplicity for both symplectic twist maps and Hamiltonian systems on cotangent bundles (Chapter 5 and Chapter 8). The results on Hamiltonian systems use techniques of decompositions of these systems into symplectic twist maps. In Chapter 7, we provide the necessary connections between these maps and Hamiltonian and Lagrangian systems, some for the first time in the literature. In particular, M. Bialy and L. Polterovitch were kind (and patient!) enough to allow me to include their proof of suspension of a symplectic twist map by an optical Hamiltonian flow. Chapter 10 presents Chaperon's proof of Arnold's conjecture on the torus, and the commonality between our methods and those of generating phases used in symplectic topology. Appendix 2 establishes the parts of Conley's theory needed in the book, including some refinements that, to my knowledge, never appeared before. For readers uncomfortable with these topics, I try to motivate Appendix 2 by a hands–on introduction to homology and Morse theory. Appendix 1 , a self contained introduction to symplectic geometry, gathers (and proves most of) the results of symplectic geometry needed in the book.

The results in this book do not make minimizing orbits their central item. In fact, they often deliberately concern systems that cannot have minimizers (non positive definite twist). However, Chapter 9 is devoted to surveying the state of affairs in the generalizations of the Aubry-Mather theory to higher dimensions, where minimizers play a fundamental role. Chapter 6, a poor substitute to a treatment that should occupy a volume on its own, surveys the theories of invariant tori (KAM theory and generalizations of Birkhoff's Graph Theorem by Bialy, Polterovitch and Herman), as well as that of splitting of separatrices.

**How to Use this Book.** Despite the survey sections interspersed throughout, this book has no encyclopedic ambitions. It aims to be an accessible platform for graduate students and researchers in mathematics and physics who want to learn about variational methods in mechanics. With this eclectic audience in mind, I strove to give entry level access to several parts of the material needed in this book. In particular, the appendices on symplectic geometry and topology are aimed at capable readers with little knowledge in these fields. In some cases, such as in the first part of the topological appendix, where a full introduction to the methods would go far beyond the scope of this book, I have chosen to sacrifice rigor, hoping to render accessible the philosophical ideas behind an often intimidating piece of theory. I have tried to make it possible for readers only interested in twist maps of the annulus or of $\mathbb{T}^n \times \mathbb{R}^n$ to read the sections pertaining to these topics with a minimum of reference to symplectic or Riemannian geometry, or to Conley's theory.

**Further Reading.** In graduate seminars at SUNY Stony Brook and UC Santa Cruz, I sometimes provided a list of complementary research articles that I or the students presented. I think the students appreciated the access to the "high summit" research, as well as the (relatively high altitude) "base camp" security of the book's material. For the 2 dimensional theory, such material could come from the topological theory of twist maps (largely absent here) as in Hall & Meyer (1991) (and its bibliography), Hedlund's theory of minimal geodesic on the torus, as revisited by Bangert (1988) or parts of the theory of renormalization in MacKay (1993) , as well as the historical articles Mather (1982) and Aubry & Le Daeron (1983). I have also been very inspired by the article of Angenent (1988), which makes good reading. For the higher dimensional symplectic twist maps , one could read some of the deep and important work of Herman on invariant tori, which I have given short shrift here (see *eg.* Herman (1990), and also Yoccoz (1992)). An excursion in KAM theory could also be a part of the reading list. In Chapter 6, I have very roughly drafted a proof of a relatively accessible KAM result from Arnold (1983). A careful exposition of its proof would be a suitable task for a graduate student (I have a fond memory of my experience doing just that as a graduate student). Very little is said here about the different types of periodic orbits one can encounter in symplectic twist maps , as well as the possible bifurcations that can take place. Kook & Meiss (1989) is a good introduction to this problem, and Arnaud (1989) gives important examples. One of the advantages of maps is that their dynamics are relatively easy to study numerically. As such, they are often used as test grounds for Hamiltonian systems. On this approach, one should consult the extensive work of Froechlé, Kook, Laskar, MacKay, and Meiss as well as the recent contribution of Tabacman and Haro. I have surveyed several generalizations of the the Aubry-Mather theory in higher dimensions in Chapter 9. Going in more depth in any of the papers surveyed there would be a good complement to that chapter. Finally, the historical Conley & Zehnder (1983) and the article of Viterbo ( 1992) could provide some depth to Chapter 10. This is by no mean an exhaustive list!

**Remarks on Style.** Finally, a few words about the style of this book. On the mathematical side, I have made a conscious choice of using local coordinates notation the most I could. This is in part to not alienate some of my physicist friends, and in part because of my personal distaste for an overly functorial notation. When I fail to check the coordinate independence of the definitions and proofs, I often urge the reader to do so. The text is accompanied by exercises, many of which form an integral part of the material and help to its understanding.

On a more typographical level, multidimensional variables, points or vectors, are usually written in slanted bold face, such as $\boldsymbol{q}$, $\boldsymbol{z}$ or $\boldsymbol{v}$. Instead of interrupting the flow of the text with formal definitions, I most often fold them in the text. A term that is defined for the first time appears in the *definition* font. The (sometimes informal) definition of the term must appear in the same paragraph. Most of the terms in the *definition* style are indexed at the end of the book. Finally, I have labeled with a star * all the chapters, sections or subsections that contain a majority of survey material - whether it be introductory or a survey of recent developments.

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