CHAPTER 9*

GENERALIZATIONS OF THE AUBRY-MATHER THEOREM

There are, strictly speaking, no *full* generalizations of the Aubry-Mather Theorem in higher dimensions: we will see in this chapter examples of fiber convex Lagrangian systems whose set of minimizers achieves only very few rotation directions. However some attempts of generalizations in higher dimensions are quite successful in what they try to achieve. In Section 45, we survey some results by de la Llave and his collaborators. Their setting is explicitly non dynamical but generalizes naturally the Frenkel-Kontorova model to functions on lattices of any dimension. They are entirely successful in proving an Aubry-Mather type theorem in this setting, as well as in some PDE cases, as well as in the context of minimal surface laminations. In Section 46, we review the work of Angenent (1990) which generalizes twist maps to a certain type of maps of $\mathbb{S}^1 \times \mathbb{R}^n$ and proves, among other results, an Aubry-Mather type theorem for these maps. In Section 47, we look at the work of MacKay & Meiss (1992) who construct higher dimensional analogs of Aubry-Mather sets in symplectic twist maps that are close to the anti-integrable limit: one where the potential term in the generating function of a standard type map dominates. In Section 48, we survey the work of Mather on minimal measures in convex Lagrangian systems. This is the closest to a generalization of the Aubry-Mather theory as one can get in the setting of general convex Lagrangian systems (as well as symplectic twist maps). We start this section by introducing the notion of minimizers and reviewing some ergodic theory. We then survey Mather's fundamental graph theorem and finish the section by pointing at the limitations of the theory. Section 49 surveys the work of Boyland and the author which shows that some of these limitations can be alleviated if one considers systems on cotangent bundles of hyperbolic manifolds.

45.* Functions on Lattices, PDE's and Minimal Surfaces

A*. Functions on Lattices

Remember from Chapter 1 that the Frenkel–Kontorova model describes configurations of interacting particles in a periodic potential. For simplicity, these configurations are assumed to be one dimensional, and the interactions only involve nearest neighbors. The resulting action function is the familiar:

$$W(\boldsymbol{x}) = \frac{1}{2}\sum_{k\in\mathbb{Z}}(x_k - x_{k+1})^2 - \sum_{k\in\mathbb{Z}} V(x_k)$$

where the potential function V has period 1. W coincides with the action function for the standard map with generating function $S(x, X) = \frac{1}{2}(X - x)^2 - V(x)$. As noted in Section 14, the variational equation $\nabla W = 0$ for this action function is

$$(-\Delta\boldsymbol{x})_k - V'(x_k) = 0$$

where $\Delta(\boldsymbol{x})_k = -2x_k + x_{k-1} + x_{k+1}$ is the discretized Laplacian. Note that the configuration $\boldsymbol{x}$ can be seen as a function $\mathbb{Z} \to \mathbb{R}$ which to the integer k makes correspond the real x_k. One obtains (see Blank (1989), Koch & al. (1994) , Candel & de la Llave (1997) , de la Llave (1999)) a natural generalization of this model, relevant to Statistical Mechanics, by asking that $\boldsymbol{x} : \mathbb{Z}^d \to \mathbb{R}$ be a function on a lattice of dimension d. We assume nearest neighbor interaction here. The energy becomes:

$$W(\boldsymbol{x}) = \frac{1}{2}\sum_{\{(k,j)\in\mathbb{Z}^d \mid\ |k-j|=1\}}(x_k - x_j)^2 - \sum_{k\in\mathbb{Z}^d} V(x_k).$$

Again V is of period 1 and the corresponding variational equation is still of the form:

$$(-\Delta\boldsymbol{x})_k - V'(x_k) = 0 \tag{45.1}$$

where $(\Delta\boldsymbol{x})_k = \sum_{|k-j|=1} x_j - 2dx_k$ is the d–dimensional discrete Laplacian. In fact, the theory in Candel & de la Llave (1997) applies to substantially more general settings, where k can belong to a set Λ on which a certain type of groups acts in a mildly prescribed way, and where the interactions involves not just nearest neighbors, but all possible pairs of particles (with some decay condition at infinity).

Remember that the solutions $\boldsymbol{x} : \mathbb{Z} \to \mathbb{R}$ found by Aubry and Mather for the Frenkel–Kontorova model are such that $|x_k - k\omega| \le \infty$. One way to express this is by saying that the graph of $\boldsymbol{x} : \mathbb{Z} \to \mathbb{R}$ is at bounded distance from a line of slope ω in $\mathbb{R} \times \mathbb{R}$. Likewise, the following generalization of the Aubry-Mather Theorem finds configurations whose graphs are at bounded distances from hyperplanes of "slopes" $\omega \in \mathbb{R}^d$. This version is taken from Candel & de la Llave (1997) :

Theorem 45.1 *For every $\omega \in \mathbb{R}^d$, there exists a solution of (45.1) such that*

$$\sup_{k \in \mathbb{Z}^d} |x_k - \omega \cdot k| < \infty.$$

The method of proof is very similar to the proof of the Aubry-Mather Theorem presented in Chapter 3. One considers the analog of CO sequences, called Birkhoff configurations by these authors. In complete analogy to the CO sequences, they satisfy:

$$x_{k+j} + l \geq x_k,\ \forall k \in \mathbb{Z}^d \quad \text{or} \quad x_{k+j} + l \leq x_k,\ \forall k \in \mathbb{Z}^d$$

The analog to the set of CO sequences of rotation number ω, which we denoted by CO_ω in Chapter 3 is:

$$\mathcal{B}_\omega = \{\boldsymbol{x} \mid \ \boldsymbol{x} \text{ is Birkhoff and } \sup_{k \in \mathbb{Z}^d} |x_k - k \cdot \omega| < \infty\}$$

In a way analogous to the proof of Theorem 15.1, one shows that the gradient flow of W (that these authors, justifiably, call the heat flow) preserves order among configurations and is suitably periodic, so that the set $\mathcal{B}_\omega$ is invariant under the flow. The same argument as in the proof of Theorem 15.1 is then used to show that W must have a critical point inside $\mathcal{B}_\omega$. So, as in the classical Aubry-Mather Theorem, one not only finds solutions that have asymptotic slope ω, but these solutions have strong order properties, expressed here in terms of non intersection: they are Birkhoff.

B*. PDE's

As Equation (45.1) suggests, the above theory smells of discretized PDE's. It is therefore not too surprising that the same kind of methods can be applied to certain PDE problems. The main ingredients necessary are some translation invariance and a heat flow that satisfies a comparison principle $u > v \Rightarrow \phi^t u > \phi^t v$, which occurs in parabolic PDE's. The method can be applied (see de la Llave (1999)) to the following PDE situations, to obtain solutions whose graphs are at bounded distance from planes with prescribed slopes, and have nonintersection properties:

$$\Delta u + V'(x,u) = 0 \tag{45.2}$$

where $V(x+e, u+\ell) = V(x,u)\ \forall x \in \mathbb{R}^d, u \in \mathbb{R}, e \in \mathbb{Z}^d, \ell \in \mathbb{Z}$.

$$\sum_{i=1}^{k} L_i^2 + V'(x,u) = 0 \tag{45.3}$$

where L_i are $\mathbb{Z}^d$ periodic vector fields satisfying Hörmander's hypoellipticity conditions and V is as in the previous case.

$$(-\Delta)^{1/2} u + V'(x,u) = 0 \tag{45.4}$$

with V as above. de la Llave (1999) also looks at the following PDE:

$$\begin{aligned} \Box\, u = u_{tt} - u_{xx} &= -V(u) + f(x,t) \\ u(x+1,t) = u(x,t+T) &= u(x,t) \end{aligned} \tag{45.5}$$

where the function f also has the periodicity:

$$f(x+1,t) = f(x,t+T) = f(x,t). \tag{45.6}$$

We say that the real number T is of constant type if its continued fraction expansion is bounded. For instance, noble numbers are of constant type.

Theorem 45.2 (de la Llave) *Let T be a number of constant type, let $f \in L^2$ satisfy (45.6) and let $V : \mathbb{R} \to \mathbb{R}$ satisfy*
(i) $0 < \alpha \le V' \le \beta$ where α is any positive number and β only depends on T (in an explicit manner)
(ii) $|V"(x)| \le K$
Then there exists a weak solution $u \in L^2$ to Equation (45.5) . Moreover, if $f \in H^r$ and $V \in C^{r+2}$ has small enough C^{r+2} norm, then there is a solution $u \in H^r$ of (45.5) which is unique in a ball in H^r around the origin.

The method of proof is different from that of the above PDE's, but still involves a variational approach.

C*. Laminations by Minimal Surfaces

Moser (1986b) proposed a generalization of the Aubry-Mather theory to certain minimizing hypersurfaces in $\mathbb{R}^n$ (according to some specific variational problem). There, Moser asks if such a result would also work when "minimizing" is understood in the classical sense of minimal area, *i.e.* minimal surfaces. Caffarelli & de la Llave (2000) answers the question positively in the following theorem:

Theorem 45.3 (Caffarelli & de la Llave) *Let g be a C^2 strictly positive metric in $\mathbb{R}^n$ invariant under integer translations. Then we can find a number M depending only on the oscillation properties of the metric such that, for every $n-1$ dimensional hyperplane Π, we can find a minimal surface Σ such that $\mathrm{Dis}(\Sigma, \Pi) \leq M$.*

Down on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, each surface Σ gives rise to a lamination, periodic if the plane Π has rational slope, "quasiperiodic" otherwise. The full result in Caffarelli & de la Llave (2000) is actually much more general, in that it deals with a large class of periodic minimization problems, those involving integrals with elliptic integrands. Moreover, as in Theorem 45.1, this result holds for manifolds whose fundamental group satisfies mild conditions, in particular manifolds of negative curvature.

Minimal Geodesics. An important special case of the above theorem is when $n = 2$. The minimal surfaces are then minimal geodesics (in the sense that they *globally* minimize length between any two of their points) for periodic metrics, *eg.* metrics on the torus $\mathbb{T}^2$. This case had been thoroughly studied by Hedlund (1932), who had obtained a similar result to the above in this case. Morse (1924) had previously treated the case of surfaces of negative

curvature. The analogy of Hedlund's work with the Aubry-Mather theorem is made clear in Bangert (1988), where he shows that both theories are part of a third one. We will see in a later section that Mather (1991b) also arrives at the same kind of conclusion, with his theory of minimal measures.

46.* Monotone Recurrence Relations

Angenent (1990) proposes a generalization of twist maps of the annulus to maps of $\mathbb{S}^1 \times \mathbb{R}^N$ which are defined by solving a recurrence relation:

$$\Delta(x_{k-l}, \dots, x_{k+m}) = 0 \tag{47.1}$$

which generalizes $\partial_2 S(x_{k-1}, x_k) + \partial_1 S(x_k, x_{k+1}) = 0$ in twist maps, where $k = l = 1$. The function Δ is required to satisfy the conditions:

a) *Monotonicity* $\Delta(x_{-l}, \dots, x_{+m})$ is a non decreasing function of all the x_k except possibly for $k = 0$. Moreover, it is strictly increasing in the variables x_{-l} and x_m.

b) *Periodicity* $\Delta(x_{k-l}, \dots, x_{k+m}) = \Delta(x_{k-l} + 1, \dots, x_{k+m} + 1)$.

c) *Coercion* $\lim_{x_l \to \pm\infty} \Delta(x_{-l}, \dots, x_m) = \lim_{x_m \to \pm\infty} \Delta(x_{-l}, \dots, x_m) = \pm\infty$.

Under these conditions, Angenent calls (47.1) a *monotone recurrence relation.* Conditions a) and c) imply that one can solve for x_{k+m} in terms of a given $(x_{k-l}, \dots, x_{k+m-1})$. Hence this defines a map $F_\Delta : (x_{k-l}, \dots, x_{k+m-1}) \mapsto (x_{k-l+1}, \dots, x_{k+m})$ from $\mathbb{R}^{l+m}$ to itself. Condition b) implies that this maps descends to a map on $\mathbb{S}^1 \times \mathbb{R}^{l+m-1}$. Hence the N above is $N = l + m - 1$.

The notion of CO configurations, rotation number and partial order on sequences etc... of Chapter 2 and Chapter 3 are still entirely valid here, since the variables x_k are 1 dimensional (Angenent also calls CO sequences Birkhoff). An interesting notion that Angenent (1990) introduces, inspired by PDE methods, is that of sub– or supersolution of the monotone recurrence relation (47.1) : $\underline{\boldsymbol{x}}$ is a *subsolution* if $\Delta(\underline{x}_{k-l}, \dots, \underline{x}_{k+m}) \le 0,\ \forall k \in \mathbb{Z}$ and a *supersolution* if $\Delta(\underline{x}_{k-l}, \dots, \underline{x}_{k+m}) \ge 0,\ \forall k \in \mathbb{Z}$.

Theorem 47.1 (Angenent) *Let $\underline{\boldsymbol{x}}, \overline{\boldsymbol{x}}$ be sub– and supersolutions respectively, which are ordered: $\underline{\boldsymbol{x}} \le \overline{\boldsymbol{x}}$. Then there is at least one solution of (47.1), say $\boldsymbol{x}$, for which $\underline{\boldsymbol{x}} \le \boldsymbol{x} \le \overline{\boldsymbol{x}}$ holds.*

Using this theorem (whose proof is simple), Angenent (1990) is able to generalize a theorem of Hall (1984), itself a generalization of the Aubry-Mather theorem: if a twist map of the annulus, which is *not necessarily* area preserving, has a (m, n)-periodic orbit, then it must have a CO (m, n)-periodic orbit. If the map is also area preserving, this implies, taking limits, the existence of CO orbits of all rotation numbers. Analogously, Angenent proves that if there is an orbit of F_Δ with rotation number $\omega \in \mathbb{R}$, then F_Δ must also have a CO orbit of rotation number ω.

Suppose that two solutions $\boldsymbol{x}$ and $\boldsymbol{w}$ of (47.1) "exchange rotation numbers" in the sense that:

$$\lim_{n\to+\infty} x_k/k \geq \omega_1 \geq \lim_{n\to-\infty} w_k/k$$

and

$$\lim_{n\to+\infty} w_k/k \leq \omega_0 \leq \lim_{n\to-\infty} x_k/k$$

holds for some $\omega_0 \leq \omega_1$. Then Angenent proves that there must be CO orbits of any rotation number $\omega \in [\omega_0, \omega_1]$. Moreover this exchange of rotation numbers condition implies chaos: the topological entropy $h_{top}(F_\Delta) > 0$, in that there is a compact invariant set semi conjugate to a Bernouilli shift. This also generalizes shadowing results of Hall (1989) and Mather (1991a). Angenent (1992) uses similar techniques to prove the following beautiful theorem:

Theorem 47.2 *Let f be a twist map of the compact annulus. Let $\rho_0 < \rho_1$ be the rotation number of f restricted to the boundaries. If the topological entropy $h_{top}(f)$ of f vanishes, then f must have an invariant circle of rotation number $\omega \in [\rho_0, \rho_1]$.*

48.* Anti–Integrable Limit

MacKay & Meiss (1992) explore the existence of Aubry-Mather sets (as well as many other possible configurations) close to the anti-integrable limit, where the potential of a standard like map becomes all powerful. Consider a family F_ϵ of symplectic twist maps of $T^*\mathbb{T}^n$ given by the generating functions:

$$S_\epsilon(\boldsymbol{q}, \boldsymbol{Q}) = \epsilon T(\boldsymbol{q}, \boldsymbol{Q}) + V(\boldsymbol{q})$$

where, for simplicity, we can assume

$$T(\boldsymbol{q}, \boldsymbol{Q}) = \frac{1}{2}(\boldsymbol{Q} - \boldsymbol{q})^2,$$

although more general T's can be considered. As usual, orbits of F_ϵ correspond to solutions of

$$\partial_2 S_\epsilon(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k) + \partial_1 S_\epsilon(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) = 0. \tag{48.1}$$

Even though F_0 is not defined, it is perfectly acceptable to set $\epsilon = 0$ in Formula (48.1) . This is called the *anti-integrable limit* , a notion that seems to have appeared independently in Aubry & Abramovici (1990) and Tangerman & Veerman (1991). The force of this concept is that the solutions of (48.1) at $\epsilon = 0$ are perfectly understood: they are simply allocations of $\boldsymbol{q}_k$ to one of the critical points of V: (48.1) is just $dV(\boldsymbol{q}_k) = 0$ when $\epsilon = 0$. If V is a Morse function, it has finitely many critical points modulo $\mathbb{Z}^n$ and they are all nondegenerate. This has the following consequence:

Theorem 48.1 (MacKay–Meiss) *Any solution $\boldsymbol{q}(0)$ of (48.1) for $\epsilon = 0$ continues to a solution $\boldsymbol{q}(\epsilon)$ when ϵ is small.*

Proof. Rewrite the infinite system of equations (48.1) in the form

$$G(\epsilon, \boldsymbol{q}) = 0$$

where $G : \mathbb{R} \times X \to (\mathbb{R}^n)^{\mathbb{Z}}$ is given by $G(\epsilon, \boldsymbol{q}) = \partial_2 S_\epsilon(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k) + \partial_1 S_\epsilon(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$ and X is the Banach space of sequences such that $\sup_k \|\boldsymbol{q}_k - \boldsymbol{q}_k(0)\| < \infty$. The Implicit Function Theorem on Banach spaces (see Lang (1983)) applies here to find, for small ϵ, a $\boldsymbol{q}(\epsilon)$ such that $G(\epsilon, \boldsymbol{q}(\epsilon)) = 0$ as long as $\frac{\partial G}{\partial \boldsymbol{q}}(0, \boldsymbol{q}(0))$ is invertible. But this is indeed the case since

$$\frac{\partial G}{\partial \boldsymbol{q}}(0, \boldsymbol{q}(0))_k = V''(\boldsymbol{q}_k(0))$$

so that $\frac{\partial G}{\partial \boldsymbol{q}}(0, \boldsymbol{q}(0))$ is an infinite block diagonal matrix with the $n \times n$ diagonal blocks $V''(\boldsymbol{q}_k(0))$ all invertible and uniformly bounded. Indeed these matrices are chosen among a finite set, since $\boldsymbol{q}_k(0)$ is necessarily a critical point of V, of which there are finitely many mod $\mathbb{Z}^n$, by the assumption that V is Morse. $\square$

One can simultaneously continue compact sets of stationary solutions from the anti-integrable limit. Such sets can be quite complicated, since the set of all stationary configurations of the anti-integrable limit can be seen as a shift on as many symbols as there are critical points. In particular, one can find invariant Cantor sets for F_ϵ. One can also get orbits with all rotation vectors $\omega \in \mathbb{R}^n$. To do so, consider the anti–integrable stationary solution $\boldsymbol{q}(0)$ which is such that $\boldsymbol{q}_0(0)$ is at some arbitrarily chosen critical point of V and

$$\boldsymbol{q}_k(0) = [k\omega] + \boldsymbol{q}_0(0),$$

where $[k\omega]$ is the integer part of the vector $k\omega$. Each $\boldsymbol{q}_k(0)$ is thus on the same critical point as $\boldsymbol{q}_0(0)$, but translated by the integer vector $[\omega]$. Since $|\boldsymbol{q}_k(0) - k\omega| < \sqrt{n}$, $\boldsymbol{q}(0)$ has rotation vector ω. Now use Theorem 48.1 to continue this to an orbit of F_ϵ with rotation vector ω. One can also continue simultaneously all anti–integrable solutions as the above with rotation vectors in a compact set: they themselves form a compact set.

Even though this seems almost too easy, the anti-integrable limit is a very useful concept in order to understand the spectrum of all possible dynamics of symplectic twist maps. It is fair to say that, to this date, the least understood cases are those that are neither close to integrable nor to anti–integrable.

49.* Mather's Theory of Minimal Measures

We now come to Mather's theory of existence and regularity of minimizers. This theory is quite general: it covers a wide class of convex Lagrangian systems on tangent bundles of arbitrary compact manifolds. Note that similar, but less developed theories were created by Bangert (1989) in the setting of minimal geodesics on compact manifolds and Katok (1992) in the setting of perturbations of integrable symplectic twist maps. There is no doubt that Mather's theory could be worked out for general symplectic twist maps. Even now, the correspondences between Lagrangian systems and symplectic twist maps given in Chapter 7 (see in particular the Bialy-Polterovitch suspension theorem 41.1) should allow an ample transfer of Mather's results to the symplectic twist maps case.

The lesson we get from Mather's work is that, yes, minimizers in general manifolds behave very much like those on the circle (the realm of the classical Aubry-Mather theory), in that they satisfy a graph property. The bad news is that minimizers may be much scarcer than in the circle case: Hedlund (1932) had already constructed a Riemannian metric on $\mathbb{T}^3$ (a setting encompassed by Mather's) which is very small along 3 non intersecting geodesics which generate $H_1(\mathbb{T}^3)$. All other minimizers of a certain length are then bound to spend a good portion of their time close to these geodesics. In particular, these three geodesics are the only possible recurrent minimizers. This limits the possible rotation vectors of minimizers to these three directions only. Bangert (1989) (geodesic setting) and Mather (1991b) (Lagrangian setting) show that, in a precise sense, this is the worst case scenario:

there should be at least as many rotation directions represented by minimizers as there are dimensions in $H_1(M, \mathbb{R})$. And, to end on an optimistic note, Levi (1997) constructs, in this worst case scenario of Hedlund's example, "shadowing" locally minimizing orbits that spend any prescribed proportion of time close to each of the minimizers. In particular, he constructs *locally minimizing* orbits of all rotation vectors.

In order to introduce Mather's theory of minimal measures, we need to define the notion of Lagrangian minimizers, similar to that of Aubry minimizers for twist maps. In passing, we note the connection between the notion of action minimizing and hyperbolicity.

A*. Lagrangian Minimizers

Throughout this section and the next, we consider time-periodic Lagrangian systems determined by a C^2-Lagrangian functions $L : TM \times \mathbb{S}^1 \to \mathbb{R}$, where M is a compact manifold given a Riemannian metric g. Remember (see Appendix 1 and Chapter 7) that extremals of the action

$$A(\gamma) = \int_a^b L(\gamma, \dot\gamma, t)dt$$

satisfies the Euler-Lagrange equations $\frac{d}{dt}\frac{\partial L}{\partial \dot x} - \frac{\partial L}{\partial x} = 0$. Using local coordinates these equations yield a first order time-periodic differential equation on TM, and thus in the standard way, a vector field on $TM \times \mathbb{S}^1$. This can be viewed as the Hamiltonian vector field corresponding to the Lagrangian system, pulled back to TM by the Legendre transformation. Since $TM \times \mathbb{S}^1$ is not compact it is possible that trajectories of this vector field are not defined for all time in $\mathbb{R}$ and thus do not fit together to give a global flow ($i.e.$ an $\mathbb{R}$-action).

When the flow does exist, it is called the *Euler-Lagrange (or E-L) flow*. The following quite general hypotheses are the setting of Mather (1991b) .

Mather's Hypotheses.

L is a C^2 function $L : TM \times \mathbb{S}^1 \to \mathbb{R}$ that satisfies:

(a) *Convexity:* $\frac{\partial^2 L}{\partial v^2}$ is positive definite.

(b) *Completeness:* The Euler-Lagrange (global) flow determined by L exists.

(c) *Superlinear:* $\frac{L(x,v,t)}{\|v\|} \to \infty \; when \; \|v\| \to +\infty$.

Mather's Hypotheses are satisfied by mechanical Lagrangians, $i.e.$ those of the form

$$L(x,v,t) = \frac{1}{2}\|v\|^2 - V(x,t),$$

where the norm is taken with respect to any Riemannian metric on the manifold. (In fact, one may allow the norm to vary with time, under some conditions, see Manẽ (1991), page 44).

Minimizers. We know that, for twist maps, orbits on Aubry-Mather sets are minimizers in the sense of Aubry. We have also seen in Chapter 6 that orbits on KAM tori are minimizers for symplectic twist maps . These are natural reasons to look for minimizers in convex Lagrangian systems. Lagrangian minimizers are defined in a way analogous to the discrete case. If $\tilde{M}$ is a covering space of M (see Appendix 2), L lifts to a real valued function (also called L) defined on $T\tilde{M} \times \mathbb{S}^1$.

A curve segment $\gamma : [a,b] \to \tilde{M}$ is called a $\tilde{M}$*-minimizing segment* or an $\tilde{M}$*-minimizer* if it minimizes the action among all absolutely continuous curves $\beta : [a,b] \to \tilde{M}$ which have the same endpoints as γ. A curve $\gamma : \mathbb{R} \to \tilde{M}$ is also called a *minimizer* if $\gamma_{|_{[a,b]}}$ is a minimizer for all $[a,b] \subset \mathbb{R}$. When the domain of definition of a curve is not explicitly given it is assumed to be $\mathbb{R}$. In practice, the two main covering spaces that we will consider are the universal cover (in next section) and the universal abelian cover (in this section, see Appendix 2 for the definitions of these covering spaces).

A fundamental theorem of *Tonelli* (see Mather (1991b) or Manẽ (1991)) implies that if L satisfies Mather's Hypotheses, then given $a < b$ and two distinct points $x_a, x_b \in \tilde{M}$ there always is a minimizer γ with $\gamma(a) = x_a$ and $\gamma(b) = x_b$. Moreover such a γ is automatically C^2 and satisfies the Euler-Lagrange equations (this uses the completeness of the E-L flow). Hence its differential $d\gamma(t) = (\gamma(t), \dot{\gamma}(t))$ yields a solution $(d\gamma(t), t)$ of the E-L flow.

B*. Ergodic Theory

Most of the material in this subsection can be found in Hasselblat & Katok (1995), where it is thoroughly developped. We start by motivating this theory by the following trivial remark: if F is a map of $T^*\mathbb{T}^n$ and $\phi(\boldsymbol{z}) = \pi(F(\boldsymbol{z})) - \pi(\boldsymbol{z})$ ($\pi : T^*\mathbb{T}^n \to \mathbb{T}^n$ is the canonical projection) then, $when\ it\ exists$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \phi(F^k(\boldsymbol{z})) = \lim_{n\to\infty} \frac{\pi(F^n(\boldsymbol{z})) - \pi(F(\boldsymbol{z}))}{n} = \rho_F(\boldsymbol{z}),$$

the rotation vector of $\boldsymbol{z}$ under F. The expression $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\phi(F^k(\boldsymbol{z}))$ for a general continuous function ϕ is called the *time average* of ϕ. Hence the rotation vector of a point, when it exists, is the time average of a specific ϕ. The relevance of this is the following:

Theorem 49.1 (Birkhoff's Ergodic Theorem) *Let $F:(X,\mu)\to(X,\mu)$ be a measure preserving transformation for a Borel measure μ on a space X, and $\phi\in L^1(X,\mu)$. Then the time average $\phi_T(\boldsymbol{z})$ of ϕ:*

$$\phi_T(\boldsymbol{z}) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\phi(F^k(\boldsymbol{z}))$$

exists for $\mu-a.e.$ $\boldsymbol{z}$. Moreover, if $\mu(X)<\infty$, $\int_X \phi_T d\mu = \int_X \phi d\mu$.

Remember that a Borel measure on a topological space is one whose sigma-algebra of measurable sets is generated by the open sets. The above theorem actually does not require the measure to be Borel, but we will assume it for the rest of this section. That F is measure preserving means $\mu(F^{-1}(A))=\mu(A)$ for any Borel subset of X. An immediate corollary of Birkhoff's theorem is (one needs to compactify $T^*\mathbb{T}^n$ with a point at ∞):

Corollary 49.2 *Let F be a volume preserving map (eg. symplectic) of $\mathbb{T}^n$. The rotation vector ρ_F is defined on a subset of full Lebesgue measure of $T^*\mathbb{T}^n$.*

It turns out that the Lebesgue measure is only one of the many measures that a symplectic map F preserves. Take $\boldsymbol{z}\in T^*\mathbb{T}^n$ to be a N-periodic point of F, for instance, and let :

$$\eta = \frac{1}{N}\sum_{k=1}^{N}\delta_{F^k(\boldsymbol{z})}$$

where the Dirac measure $\delta_{\boldsymbol{w}}$ is the (Borel) probability measure concentrated at the point $\boldsymbol{w}$ ($\delta_{\boldsymbol{w}}(A)$ is 1 if $\boldsymbol{w}\in A$ and it is 0 if not). Since $\delta_{F^k(\boldsymbol{z})}(F^{-1}(A)) = \delta_{F^{k+1}(\boldsymbol{z})}(A)$, η is invariant under F. One of the many differences between η and the Lebesgue measure is their supports. In general, the *support* of a Borel measure μ is defined as:

$$\text{Supp }\mu = \{\boldsymbol{z}\in X \mid \mu(U)>0 \quad \text{whenever} \quad \boldsymbol{z}\in U,\ U \quad \text{open} \quad \}$$

Clearly, the support of the measure η constructed above is the orbit of the periodic point $\boldsymbol{z}$, whereas the support of the Lebesgue measure is all of $T^*\mathbb{T}^n$. Hence, the support of invariant measures is another way to conceptualize invariant sets. Let $F : X \to X$ be continuous. Then the support of any F-invariant Borel measure μ is closed, F-invariant and its complement has zero μ–measure. If $\mu(X) < \infty$, Poincaré's Recurrence Theorem implies that $\mathrm{Supp}\ \mu$ is contained in the set of $F-$recurrent points. In fact, if a point $\boldsymbol{z}$ is in $\mathrm{Supp}\ \mu$ then its ω-limit set $\omega(\boldsymbol{z})$ is included in $\mathrm{Supp}\ \mu$. Hence, to find recurrent orbits in a dynamical system, as we have been doing in this book, one can look for (supports of) invariant measures.

Coming back to rotation vectors, and the measure η supported on a periodic orbit, the rotation vector $\rho_F(\boldsymbol{z})$ not only exist $\eta - a.e.$, but it is constant on $\mathrm{Supp}\ \eta$. In fact, it can easily be checked that the time average ϕ_T is constant on $\mathrm{Supp}\ \eta$ for any function $\phi \in L^1(T^*\mathbb{T}^n, \eta)$: the measure η is ergodic.

Definition 49.3 An F-invariant probability measure μ on a space X is *ergodic* if it satisfies one of the following equivalent properties:

1) Every F-invariant set has μ measure 0 or 1.

2) If $\phi \in L^1(X, \mu)$ is F-invariant then ϕ is constant $\mu - a.e.$

3) The time average ϕ_T equals the *space average* $\overset{def}{=} \int \phi d\mu$, $\mu - a.e.$

Remark 49.4 The third defining property is the one of importance in this book. It implies, in particular, that for an ergodic μ, the time averages along $\mu - a.e.$ orbits (most of which have to be in the support) are all the same. In particular, let us define the *rotation vector* $\rho(\mu)$ of a measure μ to be the μ–space average of the function $\pi(F(\boldsymbol{z})) - \pi(\boldsymbol{z})$. If μ is ergodic, $\rho(\mu)$ coincides with the rotation vector of $\mu - a.e.$ orbit of F ($i.e.$ the time average). We can still define $\rho(\mu)$ for non ergodic measures, but we loose its connection to the rotation vectors of individual orbits.

It is known (see Mañe (1987)) that, if μ is ergodic, F has an orbit in $\mathrm{Supp}\ \mu$ which is dense in that support. Hence ergodicity relates to topological transitivity. The Lebesgue measure may never be ergodic for twist maps: whenever we have a chain of elliptic islands, it comprises an invariant set which is not of full Lebesgue measure. On the other hand, twist maps do have plenty of ergodic measures. We have seen above the example of a measure η supported on periodic orbits. More generally, *Aubry-Mather sets can be defined* *as supports of ergodic measures*, pull-back of measures on $\mathbb{S}^1$ invariant under circle diffeomorphisms. Indeed, take the set $\pi(M)$ in Theorem 12.8: it is the omega limit set $\Omega(T)$ for a circle diffeomorphism T. Now, pick $x \in \Omega(T)$ and take the weak* limit μ of the probability measures $\mu_N = \frac{1}{2N-1}\sum_{-N}^{N} \delta_{T^k(x)}$: the limiting measure μ defines an ergodic measure for T, and its pull back by π is ergodic for F with support the Aubry-Mather set M [the *weak* limit* is defined by $\mu_n \xrightarrow{*} \mu$ iff $\int_X \phi\mu_n \to \int_X \phi\mu$ for all continuous ϕ].

Hence our main objects of study in this book, periodic orbits and Aubry-Mather sets, are all supports of ergodic probability measures, part of the larger set $\mathcal{M}_F$ of all F-invariant Borel probability measures.

If X is a compact metric space, it turns out that the set $\mathcal{M}$ of all Borel probability measures is convex and compact under the weak* topology. Moreover $\mathcal{M}_F$ itself is a compact and convex subset of $\mathcal{M}$ for this topology. A theorem of convex analysis (Krei-Millman) says that $\mathcal{M}_F$ is then in the convex hull of its *extreme points* : those $\mu \in \mathcal{M}_F$ which cannot be written as $t\mu_1 + (1-t)\mu_2$ for two distinct $\mu_1, \mu_2 \in \mathcal{M}_F$. Finally, the extreme points are all *ergodic* measures. We will see in the next subsection that there is a strong correspondence between the (strict) convexity of a certain projection of $\mathcal{M}_F$ and the Aubry-Mather theorem.

As we will see in next section, Mather (1991b), (1993) considers measures that are invariant under the Euler-Lagrange (E-L) flow instead of a symplectic twist map . In the light of the suspension theorem of Bialy-Polterovitch (Chapter Chapter 7), his setting encompasses a large class of symplectic twist maps. All the statements that we made above are valid for E-L flows on $T\mathbb{T}^n$ provided one compactifies $T\mathbb{T}^n$ (as Mather does) in order to use the compactness of the space of E-L-invariant probability measures.

C*. Minimal Measures

For a more detailed exposition the reader is urged to consult Mather (1991b), Manẽ (1991). There is also a very nice survey of this theory in Mather (1993).

Invariant Measures, their Action and Rotation Vector. Given a E-L invariant probability measure with compact support μ on $TM \times \mathbb{S}^1$, one can define its *rotation vector* $\rho(\mu)$ as follows: let $\beta_1, \beta_2, \ldots, \beta_n$ be a basis of $H^1(M)$ and let $\lambda_1, \ldots, \lambda_n$ be closed one-forms with $[\lambda_i] = \beta_i$ in DeRham cohomology.[17] We refer the reader uncomfortable with

[17] When homology and cohomology coefficients are unspecified they are assumed to be $\mathbb{R}$, so the notation $H_1(M)$ means $H_1(M; \mathbb{R})$, etc.

(co)homology to Appendix 2 and urge her/him to read through this section thinking of the case $M = \mathbb{T}^n$, taking $[\lambda_i] = [dx_i]$, as a basis for $H^1(\mathbb{T}^n) \simeq \mathbb{R}^n$, where $(x_1, \ldots, x_n)$ are angular coordinates on $T^*\mathbb{T}^n$. Define the i^{th} component of the rotation vector $\rho(\mu)$ as

$$\rho_i(\mu) = \int \lambda_i d\mu.$$

Note that this integral makes sense when one looks at λ_i as inducing a function from $TM \times \mathbb{S}^1$ to $\mathbb{R}$ by first projecting $TM \times \mathbb{S}^1$ onto TM,

and then treating the form as a function on TM that is linear on fibers. The rotation vector does depend on the choice of basis β_i, but because these 1–forms are closed, $\rho_i(\mu)$ does not depend on the choice of representative λ_i with $[\lambda_i] = \beta_i$. Since the rotation vector is dual to forms (with pairing $< \lambda, \mu >= \int \lambda d\mu$), it can be viewed as an element of $H_1(M)$. In the case $M = \mathbb{T}^n$, if $\gamma(0)$ is a generic point of an *ergodic* measure μ, the natural definition of *rotation vector of a curve* γ coincides with $\rho(\mu)$:

$$\rho_i(\gamma) \stackrel{def}{=} \lim_{b-a\to\infty} \frac{\tilde{\gamma}_i(b) - \tilde{\gamma}_i(a)}{b-a} = \lim_{b-a\to\infty} \frac{1}{b-a} \int_{d\gamma|_{[a,b]}} dx_i = \int dx_i d\mu = \rho_i(\mu)$$

where $\tilde{\gamma}$ is a lift of γ to $\mathbb{R}^n$ and the third equality uses the Ergodic Theorem (again, dx_i is seen as a function $TM \times \mathbb{S}^1 \to \mathbb{R}$). This prompts the following formula for the (i^{th} coordinate of the) rotation vector of a curve $\gamma : \mathbb{R} \to M$ for a general manifold M:

$$\rho_i(\gamma) = \lim_{b-a\to\infty} \frac{1}{b-a} \int_{d\gamma|_{[a,b]}} \lambda_i,$$

if the limit exists. As before, if $\gamma(0)$ is a generic point for an *ergodic* measure μ, $\rho(\gamma)$ exists and coincides with $\rho(\mu)$ (and this is not necessarily the case for non ergodic measures). Next we define the *average action* of a E-L invariant probability on $TM \times \mathbb{S}^1$:

$$A(\mu) = \int L d\mu,$$

i.e. the space average of L which, when μ is ergodic, equals the time average along μ-a.e. orbit γ:

$$A(\mu) = \lim_{b-a\to\infty} \frac{1}{b-a} \int_a^b L(\gamma, \dot{\gamma}) dt, \qquad \mu - a.e. \text{ orbit } \gamma.$$

The Set of Minimal Measures and the Function β. The set of E–L invariant probability measures, denoted by $\mathcal{M}_L$, is a convex set in the vector space of all measures, as we have seen in the previous subsection (it is also compact for the weak* topology if, as Mather does, one compactifies TM) and the extreme points of $\mathcal{M}_L$ are the ergodic measures (see Mañe (1987)). Consider the map $\mathcal{M}_L \to H_1(M) \times \mathbb{R}$ given by:

$$\mu \mapsto (\rho(\mu), A(\mu)) \,.$$

This map is trivially linear and hence maps $\mathcal{M}_L$ to a convex set U_L whose extreme points are images of extreme points of $\mathcal{M}_L$, $i.e.$ images of ergodic measures. Mather shows, by taking limits of measures supported on long minimizers representing rational homology classes, that for each ω, there exists an invariant (but not necessarily ergodic) measure μ such that $\rho(\mu) = \omega$ and $A(\mu) < \infty$.[18] Since L is bounded below, the action coordinate is bounded below on U_L. Hence we can define a map $\beta : H_1(M) \to \mathbb{R}$ by

$$\beta(\omega) = \inf\{A(\mu) \mid \ \mu \in \mathcal{M}_L, \rho(\mu) = \omega\},$$

which is bounded below and convex: the graph of β is the boundary of U_L. We say that a probability measure $\mu \in \mathcal{M}_L$ is a *minimal measure* if the point $(\rho(\mu), A(\mu))$ is on the graph of β. In other words, a measure μ with $\rho(\mu) = \omega$ is a minimal measure *iff* it minimizes the average action subject to the constraint that it have rotation vector ω. By construction, an extreme point $(\omega, \beta(\omega))$ of $graph(\beta)$ corresponds to at least one minimal ergodic measure of rotation vector ω. It turns out that if μ is minimal, μ-a.e. orbit lifts to a E-L minimizer in the universal abelian cover $\overline{M}$ of M (whose deck transformation group is $H_1(M; \mathbb{Z})/torsion$, see Appendix 2). Conversely, if μ is an ergodic probability measure whose support consists of $\overline{M}$-minimizers, then μ is a minimal measure.

Hence, each time we prove the existence of an extreme point $(\omega, \beta(\omega))$, we find at least one recurrent orbit of rotation vector ω which is a $\overline{M}$-minimizer.

Another important property of β is that it is superlinear, $i.e.$ $\frac{\beta(x)}{\|x\|} \to \infty$ when $\|x\| \to \infty$. We motivate this in the simple case where $L = \frac{1}{2}\|\dot{x}\|^2 - V(x)$ and $\|\cdot\|$ comes from the Euclidean metric on the torus. If μ is any invariant probability measure, then

[18] The impatient reader may be tempted to proclaim, from this fact, the existence of orbits of all rotation vectors. Alas, as we noted in Remark 49.4, we can guarantee that the rotation vector of orbits in the support of a measure μ are equal to $\rho(\mu)$ only when μ is ergodic

$$
\begin{aligned}
A(\mu) &= \int L \, d\mu \geq \int \left(\frac{\|\dot{x}\|^2}{2} - V_{max} \right) d\mu \\
&\geq \frac{1}{2} \left| \int \dot{x} \, d\mu \right|^2 - V_{max} \qquad (49.1) \\
&= \frac{1}{2} |\rho(\mu)|^2 - V_{max},
\end{aligned}
$$

where we used the Cauchy-Schwartz inequality for the second inequality. We see that in this particular but important case, β grows at least quadratically with the rotation vector. The superlinearity of β implies the existence of many extreme points for $graph(\beta)$ (although in most cases still too few, as we will see in the next subsection), as we explain now.

Countably Many Minimal Measures: Is This Enough? As the boundary of the convex set U_L, the graph of β is made of flat faces, or *linear domains*. Each of these linear domains, which we denote by S_c, is a convex set, intersection of U_L with a supporting hyperplane of U_L of "slope" c. [Since c acts linearly on homology classes ω to give the equation $c \cdot \omega = a$ of the supporting hyperplane, it can be seen as an element of first cohomology.] Let X_c be the projection on $H_1(M)$ of S_c. The sets X_c are convex domains which tile the space $H_1(M)$. Now the growth condition on β implies that

its graph cannot have a linear domain going to infinity: the sets X_c must be compact. Extreme points of X_c are projections of extreme points of S_c, themselves extreme points of U_L. Hence there are infinitely many such extreme points, and infinitely many outside any compact set. Their convex hull is

$H_1(M)$, and in particular, they must span $H_1(M)$ as a vector space. Since these extreme points are the rotation vectors of minimal ergodic measures, we have found:

Theorem 49.5 *There exist at least countably many minimal ergodic measures and at least $n = \dim H_1(M)$ of them with distinct rotation directions.*

In particular there are at least n rotation directions represented by minimal measures for a E-L flow on $T^*\mathbb{T}^n$. We will see in Hedlund's example that this lower bound *is* attained by some systems. Finally, the *generalized Mather sets* are defined as

$$M_c = Support(\mathcal{M}_c),$$

where $\mathcal{M}_c$ is the set of all minimal measures whose rotation vectors lies in X_c. Let $\pi : TM \times \mathbb{S}^1 \to M \times \mathbb{S}^1$ denote the projection. The main result in Mather (1991b) is the following theorem:

Theorem 49.6 (Mather's Lipschitz Graph Theorem) *For all $c \in H^1(M)$, M_c is a compact, non-empty subset of $TM \times \mathbb{S}^1$. The restriction of π to M_c is injective. The inverse mapping $\pi^{-1} : \pi(M_c) \to M_c$ is Lipschitz.*

In the case $M = \mathbb{T}^n$, Mather proves that, when they exist, KAM tori coincide with the sets M_c (see also Katok (1992) for some related results in the symplectic twist maps context). The proof of the Graph Theorem (see Mather (1991b) or Manẽ (1991)), which is quite involved, uses a curve shortening argument: if two minimizing curves in $\pi(M_c)$ were too close to crossing transversally, one could "cut corners" and, because of recurrence, construct a closed curve in M_c with lesser action than the minimal action of measures in M_c, a contradiction. This argument by surgery is reminiscent of the proof of Aubry's Fundamental Lemma in Chapter 2.

The Autonomous Case. An important special case is that of $autonomous$ systems ($i.e.$ with time independent L). In this case, one can discard the time component and view M_c as a compact subset of TM. Mather's theorem implies that M_c is a Lipschitz graph for the projection $\pi : TM \to M$. To see this, suppose that two curves $x(t)$ and $y(t)$ in $\pi(M_c)$ have $x(0) = y(s)$ for some s. Mather's theorem rules out immediately the

possibility that s is an integer, unless $x = y$ is a periodic orbit. For a general s, consider the curve $z(t) = y(t+s)$. Then, $\dot{z}(t) = \dot{y}(t+s)$ and, by time-invariance of the Lagrangian, $(z(t), \dot{z}(t))$ is a solution of the E-L flow. It has same average action and rotation vector as $(y, \dot{y})$ and hence it is also in M_c. But then $z(0) = x(0)$ is impossible, by Mather's theorem, unless $\dot{z}(0) = \dot{y}(s) = \dot{x}(0)$ and thus, by uniqueness of solutions of ODEs, $x(t) = y(t+s)$.

The Symplectic Twist Map Case. By using Theorem 41.1, one can translate the results of Mather to the realm of symplectic twist maps (see Exercise 49.7) and deduce the existence of many invariant sets that are graphs over the base and are made of minimizers. As noted before, one could also redo all of Mather's theory in the setting of symplectic twist maps (see Katok (1992), who considered the near integrable case).

D*. Examples and Counterexamples

Recovering Past Results. When Mather's function β (see previous section) is strictly convex, each point on $graph(\beta)$ is an extreme point and there are ergodic minimal measures (and hence minimal orbits) of all rotation vectors. One can prove that this is true when $M = \mathbb{S}^1$, and Mather (1991b) shows how his Lipschitz Graph Theorem implies the classical Aubry-Mather Theorem, by taking a E-L flow that suspends the twist map. The fact that M_c is a graph nicely translates into the fact that orbits in an Aubry-Mather set are cyclically ordered: as pointed out by Hall (1984), the CO property corresponds to trivial braiding of the suspended orbit, itself guaranteed by the graph property.

The graph of β is also strictly convex when L is a Riemannian metric on $\mathbb{T}^2$, and hence there are minimal geodesics of all rotation vectors for any metric on the torus. This was known by Hedlund (1932), who had basically worked out the same results as Aubry and Mather in that setting, albeit in a different language. [See Bangert (1988) for a unified approach of the two theories]. Hence one could hope, as a generalization of the Aubry-Mather theorem, that β is strictly convex for any Lagrangian systems satisfying Mather's hypotheses. This statement is false as we will see in the following examples.

Examples of Gaps in the Rotation Vector Spectrum of Minimizers for Lagrangian on $\mathbb{T}^2$. Take $L : T\mathbb{T}^2 \to \mathbb{R}$, given by $L(x, \dot{x}) = \|\dot{x} - X\|^2$ where X is a vector field on $\mathbb{T}^2$. The integral curves x of X are automatically E-L minimizers since $L \equiv 0$ on these curves. Manẽ (1991) chooses the vector field X to be a (constant) vector field of irrational slope multiplied by a carefully chosen function on the torus which is zero at exactly one point q. The integral flow of X has the rest point $q(t) = q$, and all the other solutions are dense on the torus. The flow of X (and its lift to $T\mathbb{T}^2$ by the differential) has exactly two ergodic measures: one is the Dirac measure supported on $(q, 0)$, with zero rotation vector, the other is equivalent to the Lebesgue measure on $\mathbb{T}^2$ and has nonzero rotation vector, say ω. Mañe checks that $\beta^{-1}(0)$ (trivially always an X_c) is the interval $\{\lambda\, \omega \mid \ \lambda \in [0, 1]\}$, and that no ergodic measure has a rotation vectors strictly inside this interval. Thus the Mather set M_0 is the union of the supports of the two above measures.

Boyland & Golé (1996a) give an example of an autonomous *mechanical* Lagrangian on $\mathbb{T}^2$ which displays a similar phenomenon, although we also show in that paper that all autonomous Lagrangian systems satisfying Mather's Hypothesis do have minimizers of all rotation *directions*. We also give in this paper a very precise description of the β function for such systems and show that the support of minimal ergodic measures have to be either a point, a closed curve, a suspension of a Denjoy Cantor set or the whole torus.

Hedlund-Bangert's Counterexamples. Consider in $\mathbb{R}^3$ the three nonintersecting lines given by the x-axis, the y-axis translated by $(0,0,1/2)$ and the z-axis translated by $(1/2,1/2,0)$. Construct a $\mathbb{Z}^3$- lattice of nonintersecting lines by translating each one of these three lines by integer vectors. Take a metric in $\mathbb{R}^3$ which is the Euclidean metric everywhere except in small, nonintersecting tubes around each of the axes in the lattice. In these tubes, multiply the Euclidean metric by a positive function λ which is 1 on the boundary and attains its (arbitrarily small) minimum along the points in the center of the tubes, $i.e.$ at the axes of the lattice. Because the construction is $\mathbb{Z}^3$ periodic, this metric induces a Riemannian metric on $\mathbb{T}^3$. One can show (Bangert (1989)), if λ is taken sufficiently small, that a minimal geodesic (which is a E-L minimizer in our context) can make at most three jumps between tubes. In particular, a recurrent E-L minimizer has to be one of the three disjoint periodic orbits which are the projection of the axes of the lattice. Thus there are only three rotation directions that minimizers can take in this example, or six if one counts positive and negative orientations. In terms of Mather's theory, the level sets of the function β in $\mathbb{R}^3 = H_1(\mathbb{T}^3)$ are octahedrons with vertices $(\pm a,0,0),(0,\pm a,0),(0,0,\pm a)$ (we assume here that the function λ is the same around each of the tubes). Since we are in the case of a metric, one can check that β is quadratic when restricted to a line through the origin (a minimizer of rotation vector $a\omega$ is a reparameterization of a minimizer of rotation ω). Hence a set S_c is either a face, an edge or a vertex of some level set $\{\beta = b\}$, and the corresponding M_c is, respectively, the union of three, two (parameterized at same speed) or one of the minimal periodic orbits one gets by projecting the disjoint axes. Note that, instead of the function β of Mather, Bangert uses the *stable norm.* Mather's function β is a generalization of that norm.

Levi's Counter–Counterexample. It is important to note that the nonexistence of minimizers of a certain rotation vector ω does not mean that there are no orbits of the E-L flow that have rotation vector ω. For example, Levi (1997) has shown the existence of orbits of all rotation vectors in the Hedlund example. He constructs, using some broken geodesic methods, *local* minimizers shadowing any curve made of segments (of sufficient length) of the minimizing axes and jumps between the axes. This makes for extremely rich, chaotic dynamics.

Further Developments of Mather's Theory of Minimal Measures. As of this writing, a number of mathematicians are very active in research on minimizers in convex Lagrangian. Foremost is the group of young and talented researchers which formed in South America and Mexico around former students of the late Mañe. This group has solved many problems posed by Mañe soon before his death (see Mañe (1996a) and Mañe (1996b)), mainly on autonomous Lagrangian systems. Look for the names of Carneiro, Contreras, Delgado, Iturriaga, Paternain, Sanchez-Morgado and more. Recently this group, together with K.F. Siburg, has made strides in bridging this theory with the more geometric point of view of symplectic topologists: the minimal action function is related to symplectic capacities and Hofer's energy (see Siburg (1998), Iturriaga & Sánchez-Morgado (2000a)). Fathi (1997) recovers some of Mather's theory and creates ties with the KAM theory using super and sub solutions of Hamilton-Jacobi's equations. Iturriaga & Sánchez-Morgado (2000b), Contreras (2000) continue this work.

As noted in Chapter 6, Mather has recently used the theory of minimal measures, together with some hyperbolic methods to prove the existence of unbounded orbits in Lagrangian systems, a great leap in the general problem of the so-called Arnold diffusion (see Delshams, de la Llave & Seara (2000)). Hence the theory has gone far beyond the task of generalizing the Aubry-Mather theory: it has given mathematicians new tools to study the global dynamical properties of Lagrangian systems.

Exercise 49.7 Find hypotheses on the generating function of an symplectic twist map F which translate to Mather's hypotheses for the Lagrangian that suspends F (*Hint.* You may want to include Bialy and Polterovitch's conditions of Theorem 41.1 for F to have a convex suspension. Note that completeness of the flow is for free: F is defined everywhere.)

50.* The Case Of Hyperbolic Manifolds

We start this section with another counterexample to the strict convexity of Mather's β function, and hence to the existence of orbits of all rotation vectors. The setting is that of a metric on the two-holed torus, the simplest example of a compact hyperbolic manifold. However, we finish the section on a positive note, by quoting a result of Boyland & Golé (1996b), in which we introduce another definition of rotation vector suited to hyperbolic manifolds and show the existence of minimal orbits of all rotation directions for a large class of Lagrangian systems on hyperbolic manifolds.

A*. Hyperbolic Counterexample

Take the metric of constant negative curvature on the surface of genus 2 (the two-holed torus) which has a long neck between the two holes (see Figure 50.1). A minimizer here is a minimizing geodesic for the hyperbolic metric. With a and b as shown, the minimal measure for the homology class $[a] + [b]$ is a linear combination of the ergodic

measures supported on Γ_a and Γ_b, where Γ_a and Γ_b are the closed geodesics in the homotopy classes of a and b, respectively. Indeed, Γ_a and Γ_b must "go around" the same holes as a and b, and any closed curve that crosses the neck will be longer than the sum of the lengths of Γ_a and Γ_b. Hence $([a] + [b], \beta([a] + [b]))$ cannot be an extreme point of $graph(\beta)$.

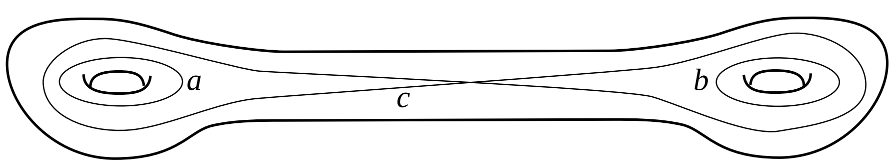

Fig. 50.1. The surface of genus two and the loops a and b. No minimal measure with rotation vector $[a] + [b]$ can have support passing through the long neck. In particular, a curve in the homotopy class of c cannot yield a minimizer in the abelian cover.

B*. All Rotation Directions in Hyperbolic Manifolds

As the previous example shows, the notion of minimizers in the abelian cover is maybe too restrictive, as it rules out many geodesics. Instead of working on the universal *abelian* cover, we work in the universal cover and define minimizers and rotation vectors with respect to that cover.

All manifolds of dimension n which admit a hyperbolic metric of constant negative curvature have the Poincaré n–disk as universal covering space $\mathbb{H}^n$. Hence a hyperbolic manifold M is the quotient $\mathbb{H}^n/\pi_1(M)$ where $\pi_1(M)$ acts on $\mathbb{H}^n$ as the group of deck transformations. To visualize $\mathbb{H}^n$, assume $n = 2$, which covers any orientable surface of genus greater or equal to two. One model for $\mathbb{H}^2$ is the usual Euclidean unit disk which is given the hyperbolic metric $\frac{dx^2+dy^2}{1-(x^2+y^2)}$. The ratio between the corresponding hyperbolic distance and the euclidean one tends exponentially to ∞ as points approach the boundary of the disk. Geodesics for the hyperbolic metric are arcs of (Euclidean) circles perpendicular to the boundary $\partial\mathbb{H}^2$ of the disk.

The minimizers we consider in this section lift to curves in the *universal cover* which minimize the action between any two of their points. We also assume that the Lagrangian L satisfies Mather's Hypotheses (time periodic C^2 function with (a) fiber convexity, (b) completeness of the E–L flow) except that we replace his condition (c) of superlinearity by one of superquadraticity:

(c') *superquadraticity*: There exists a $C > 0$ such that $L(x, v, t) \geq C\,\|v\|^2$.

This, again, is satisfied by mechanical systems. Note that, without loss of generality, one can assume the Lagrangian L to be positive: being convex, it is bounded below, and adding a constant to L does not change the E–L solutions. We now state the two theorems that appear in Boyland & Golé (1996b). The first one finds minimizing solutions near any given geodesic:

Theorem 50.1 (Boyland–Golé) *Let (M, g) be a closed hyperbolic manifold. Given a Lagrangian L which satisfies Hypotheses (a), (b), (c'), there are sequences k_i, κ_i, T_i in $\mathbb{R}^+$ depending only on L, with k_i increasing to infinity, such that, for any hyperbolic geodesic $\Gamma_0 \subset \mathbb{H}^n = \tilde{M}$ (for the lifted metric), there are minimizers $\gamma_i : \mathbb{R} \to \tilde{M}$ with $dist(\gamma_i, \Gamma_0) \leq \kappa_i$, $\gamma_i(\pm\infty) = \Gamma_0(\pm\infty)$, and $k_i \leq \frac{1}{d-c} dist(\gamma_i(d), \gamma_i(c)) \leq k_{i+1}$ whenever $d - c \geq T_i$.*

The following theorem can be viewed as a (weak) generalization of the Aubry-Mather theorem for convex Lagrangian system on compact hyperbolic manifolds. Remember that, since the geodesic flow is the solution of the autonomous Lagrangian system with corresponding Hamiltonian $\frac{1}{2}\|\boldsymbol{p}\|^2$, the energy levels $\|\boldsymbol{v}\| = c$ are all invariant sphere bundles that foliate the tangent bundle. In the case of $M = \mathbb{S}^1$ with the Euclidean metric, these energy levels are pairs of flat invariant circles of the completely integrable map. Similarly to the Aubry-Mather theorem which states that traces of these invariant circles remain in the guises of invariant circles or Cantor sets whose dynamics is (semi)conjugate to circle homeomorphisms, the following theorem proves the existence of E–L invariant sets that are semiconjugate to the geodesic flow on these sphere bundles. The reason that we call this a "weak" generalization of the Aubry-Mather theorem is that we can only guarantee the existence countably many of these E-L invariant sets. Hence the geodesic flow is "weakly" topologically stable.

Theorem 50.2 (Boyland–Golé) *Let (M, g) be a closed hyperbolic manifold with geodesic flow g_t. Given a Lagrangian L which satisfies Hypotheses (a), (b), (c') with E-L flow ϕ_t, there exists sequences k_i and T_i with k_i increasing to infinity, and a family of compact, ϕ_t-invariant sets $X_i \subset TM$ so that for all i, (X_i, ϕ_t) is semiconjugate to (T_1M, g_t) and $k_i \leq \frac{1}{T} dist(\phi_T(\boldsymbol{x}), \phi_0(\boldsymbol{x})) \leq k_{i+1}$, whenever $T \geq T_i$ and $\boldsymbol{x} \in X_i$.*

The key to these theorems is that, for any Lagrangian systems on a compact manifold satisfying the properties a), b), c'), we show that E-L solutions are quasi-geodesics, in the sense of Gromov. We then use the property that, in hyperbolic manifolds, quasi-geodesics are uniformly close to geodesics.

A New Definition of Rotation Vector in Hyperbolic Manifolds. We now interpret Theorem 50.1 as saying that there exist minimizers of all rotation directions, with a new definition of this term valid only for hyperbolic manifolds. Let us first reinterpret the classical notion of rotation vector on $T^*\mathbb{T}^n$ geometrically: a curve γ on $\mathbb{T}^n$ has rotation vector $\boldsymbol{v} \in \mathbb{R}^n$ if its lift $\tilde{\gamma}$ in the universal cover $\mathbb{R}^n$ is "asymptotically parallel" to the straight line supporting $\boldsymbol{v}$ and if the average of $\|\dot{\gamma}(t)\|$ over all $t \in \mathbb{R}$ is equal to $\|\boldsymbol{v}\|$ (we let the reader make these statement precise and rigorous). Now given two points on $\partial\mathbb{H}^2$, there is exactly one geodesic Γ_0 that goes to the first as $t \to -\infty$, to the other one as $t \to +\infty$. We can declare a curve γ to be asymptotically parallel to Γ_0 i*ff* γ and Γ_0 have same endpoints. This will insure that points of γ are always at a bounded hyperbolic distance from Γ_0. We also declare that the rotation vector exists i*ff* γ has the same endpoints at $\pm\infty$ as a geodesic Γ_0, and if the average $|\rho(\gamma)|$ of $\|\dot\gamma\|$ over $t \in \mathbb{R}$ exists, and we define the *rotation vector* to be the pair $\rho(\gamma) = (\Gamma_0, |\rho(\gamma)|)$ (average direction and average speed). In that language, Theorem 50.1 states that, given any geodesic Γ, there are infinitely many E-L minimizers with Γ as a rotation direction.

The naive definition of rotation vector that we just outlined has some major flaws:

1. $\rho(\gamma)$ (if it exists) does not belong to a linear space.

2. Two lifts of the same curve $will$ have different rotation vectors.

3. Rotation direction is not constant $\mu - a.e.$ for many ergodic measures for the geodesic flow.

To remedy that, let $\pi_1(M)$, seen as deck transformation group, act $on\ geodesics$ in $\mathbb{H}^2$ and declare that two geodesics are parallel iff they belong to the closure of the same $\pi_1(M)$-orbit (of geodesics). Consider the set of tangent vectors at all points of all the geodesics in the closure of a $\pi_1(M)$ orbit. This forms a closed subset of the unit tangent bundle of $\mathbb{H}^2$. The projection by the differential of the covering map of this set on the unit tangent bundle of M is the support of a measure μ which is invariant under the geodesic flow. Because of this, Boyland (1996) defines the rotation direction of a curve to be a measure invariant under the geodesic flow, weak* limit of measures supported by geodesics joining two points of the curve. This rotation vector being defined through ergodic theory, it is constant $\mu - a.e.$ for any E-L ergodic μ. Theorem 50.2 implies the existence of minimizer of all rotation directions, in this new, "homotopy", sense of the word.

Note that there are many more such "homotopy" directions than there are "homology" directions. For instance the "long neck" metric of Figure 50.1 has no homology minimizer with rotation direction c, as argued in the previous subsection, but it will have infinitely many homotopy minimizers with that direction.

On the negative side, the counterexamples of Manẽ (1991) and Boyland & Golé (1996a) on $\mathbb{T}^2$ as reviewed in the previous section, where gaps in the rotation spectrum are found, probably have counterparts on hyperbolic manifolds, even with our new definition of rotation vector. Thus, we think there is little chance to prove the existence of $global$ minimizers of all rotation vectors, even on these manifolds. However there should be good chances to find *local* minimizers of all rotation vectors, as we discuss in the next section.

What we hoped to show in this section is that the universal cover is a more natural setting than the *abelian* universal cover when studying Lagrangian minimizers on hyperbolic manifolds.

51.* Concluding Remarks

What, in the end, are the chances of finding orbits of all rotation vectors for symplectic twist maps or Lagrangian system, in say, $T^*\mathbb{T}^n$? Previous attempts at this problem yielded incomplete results. Bernstein & Katok (1987) "almost" found, for minimizing periodic orbits of symplectic twist maps close to integrable, some uniform modulus of continuity, which they hoped would unable them to take limits and get orbits with the limiting rotation vectors. In my thesis, I hoped that proving some regularity of the ghost tori (invariant set for the gradient flow of the periodic action) might enable one to do the same. This is how ghost circles came about.

One thing is clear: one cannot hope for *global* minimizers to achieve all possible rotation vectors. However, the shadowing methods to construct local minimizers of all rotation vectors of Levi (1997) on the Hedlund counterexamples indicate a possible approach to the general case. The recent work of Mather on existence of unbounded orbits (see Delshams, de la Llave & Seara (2000) , Section 49 and the end of Chapter 6), also shows that, for general systems, hyperbolic and variational techniques can combine powerfully to construct orbits shadowing successive minimizers. One possibility to attack the problem of existence of orbits of all rotation vector would be to try to construct, in a manner analogous to Levi (1997), orbits shadowing the different supports of the ergodic measures which are extreme points of one generalized Mather set $\mathcal{M}_c$. Doing so, one may manage to "fill in" the corresponding set of rotation vectors X_c with rotation vectors of actual orbits, may they be local minimizers.