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PERIODIC ORBITS FOR SYMPLECTIC TWIST MAPS OF T*T^n

27. Presentation of the Results

In this Chapter, we give some results on the existence of multiple periodic orbits of different rotation vectors for symplectic twist map of $T^*\mathbb{T}^n$. The introduction in Appendix 2 of new topological tools yields an improvement on the results of Golé (1989) and (1991), as well as simplifications and unification in the proofs. For the sake of fluidity, some of the settings and arguments are repeated from the two dimensional case.

A. Periodic Orbits and Rotation Vectors

Similarly to the case $n = 1$, a point $(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{R}^{2n}$ is called a $\boldsymbol{m}, d$*–periodic point* for the lift F of a map f of $T^*\mathbb{T}^n$ if

$$F^d(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{p})$$

where $\boldsymbol{m} \in \mathbb{Z}^n$ and $d \in \mathbb{Z}^+$. The rational vector $\dfrac{\boldsymbol{m}}{d}$ is called the *rotation vector* of the orbit of $(\boldsymbol{q}, \boldsymbol{p})$. We will say that $\boldsymbol{m}$ and d are *relatively prime* when d is relatively prime with at least one of the components of the vector $\boldsymbol{m}$. In general, when it exists, the *rotation vector of a sequence* $\overline{\boldsymbol{q}} = \{\boldsymbol{q}_k\}_{k\in\mathbb{Z}} \in (\mathbb{R}^n)^{\mathbb{Z}}$ is given by the limit:

$$\rho(\overline{\boldsymbol{q}}) = \lim_{k\to\pm\infty} \frac{\boldsymbol{q}_k}{k}.$$

B. Theorems of Existence of Periodic Orbits

The maps that we consider here satisfy either one of the following assumptions of coercion or asymptotic linearity. In both cases, we assume that our map is a composition of symplectic twist maps : $F = F_N \circ \ldots \circ F_1$, where each F_k is the lift of a symplectic twist map of $T^*\mathbb{T}^n$, with generating function S_k satisfying either:

Coercion

(27.1) $$\lim_{\|\boldsymbol{Q}-\boldsymbol{q}\|\to\infty} S_k(\boldsymbol{q},\boldsymbol{Q}) \to +\infty$$

or:

Asymptotic Linearity

$$S_k(\boldsymbol{q},\boldsymbol{Q}) = \frac{1}{2}\langle A_k(\boldsymbol{Q}-\boldsymbol{q}),(\boldsymbol{Q}-\boldsymbol{q})\rangle + R_k(\boldsymbol{q},\boldsymbol{Q})$$

with:

(27.2) (a) $$A_k = A_k^t,\ \det A_k \neq 0$$

(27.2) (b) $$\det \sum_1^N A_k^{-1} \neq 0$$

(27.2) (c) $$\lim_{\|\boldsymbol{Q}-\boldsymbol{q}\|\to\infty} \frac{\nabla R_k(\boldsymbol{q},\boldsymbol{Q})}{\|\boldsymbol{Q}-\boldsymbol{q}\|} = 0.$$

Equivalently:

$$F_k(\boldsymbol{q},\boldsymbol{p}) = (\boldsymbol{q} + A_k^{-1}\boldsymbol{p} + \Theta(\boldsymbol{q},\boldsymbol{p}),\ \boldsymbol{p} + \Upsilon(\boldsymbol{q},\boldsymbol{p}))$$

with (27.2) (a) and (b) holding for A_k and:

(27.2) (c') $$\lim_{\|\boldsymbol{p}\|\to\infty} \frac{\Theta(\boldsymbol{q},\boldsymbol{p})}{\|\boldsymbol{p}\|} = \lim_{\|\boldsymbol{p}\|\to\infty} \frac{\Upsilon(\boldsymbol{q},\boldsymbol{p})}{\|\boldsymbol{p}\|} = 0$$

Theorem 27.1 *Let $F = F_N \circ \ldots \circ F_1$ be a finite composition of symplectic twist maps F_k of $T^*\mathbb{T}^n$ satisfying either the convexity condition (27.1) or the asymptotic condition (27.2). Then, for each relatively prime $(\boldsymbol{m}, d) \in \mathbb{Z}^n \times \mathbb{Z}^+$, F has at least $n+1$ periodic orbits of type $\boldsymbol{m}, d$. It has at least 2^n of them when they are all non–degenerate.*

A periodic orbit is called *nondegenerate* when the composition of the differential of the map along the orbit has no eigenvalue equal to 1, see Section 29.

Outline Of The Proof. In the coercive case, we start by finding a minimum for the discrete action function W, sum of generating functions. This minimum is given by the coercion of S and its periodicity. The multiplicity is given by Morse-Conley theory on an adequately chosen sublevel set $\{W \leq C\}$.

The case with the asymptotically linear condition is a relatively easy consequence of Proposition 64.6 of Appendix 2 (first published here), which is really the technical heart of the proof. We only have to prove that the action function W on the appropriate quotient space of sequences is indeed quadratic at infinity in the sense required by that proposition.

C. Comments on the Asymptotic Conditions

Coercion vs. Convexity. As in the case of dimension 2, the convexity of the generating functions implies the coercion condition (27.1) . Namely:

Lemma 27.2 *Let S be the generating function of a symplectic twist map satisfying the following convexity condition:*

$$\langle \partial_{12} S(\boldsymbol{q},\boldsymbol{Q})\boldsymbol{v},\boldsymbol{v}\rangle \leq -a\,\|\boldsymbol{v}\|^2, \quad \forall \boldsymbol{q},\boldsymbol{Q},\boldsymbol{v}\in \mathbb{R}^n, k\in\{1,\ldots,N\}. \tag{27.3}$$

Then S is coercive. More precisely, there are a real number α and positive real numbers β and γ such that:

$$S(\boldsymbol{q},\boldsymbol{Q}) \geq \alpha - \beta\,\|\boldsymbol{q}-\boldsymbol{Q}\| + \gamma\,\|\boldsymbol{q}-\boldsymbol{Q}\|^2 . \tag{27.4}$$

The proof is identical to that of Proposition 40 of Chapter 2. The convexity condition (27.3) can be seen directly on the map. Indeed, in Proposition 25.5, we derived $\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{p}}(\boldsymbol{q},\boldsymbol{p}) = -\left(\partial_{12}S(\boldsymbol{q},\boldsymbol{Q})\right)^{-1}$, by implicit differentiation of $\boldsymbol{p} = -\partial_1 S(\boldsymbol{q},\boldsymbol{Q})$. The convexity condition (27.3) thus translates to:

$$F(\boldsymbol{q},\boldsymbol{p}) = (\boldsymbol{Q},\boldsymbol{P}) \qquad \text{and} \qquad \left\langle \left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{p}}\right)^{-1} \boldsymbol{v},\boldsymbol{v}\right\rangle \geq a\,\|\boldsymbol{v}\|^2, \quad \forall \boldsymbol{v}\in\mathbb{R}^n. \tag{27.5}$$

uniformly in $(\boldsymbol{q}, \boldsymbol{p})$. Note that Condition (27.5) means that F has bounded, positive definite twist. MacKay & al. (1989) imposed this condition in their definition of symplectic twist maps, a terminology that we have taken from them. Remember that Proposition 25.3 in Chapter 4 shows that the bounded twist condition (27.5) implies the global twist condition.

About Asymptotic Linearity. The most important feature of Condition (27.2) is that each A_k is *not necessarily positive definite*, but only a nondegenerate symmetric matrix. In particular, *no coercion* on S is assumed here and the function W will in general *not* have a minimum. This is what Herman (1990) called the *indefinite case*.

Note that if we set $R_k = 0$ in S_k, we obtain a quadratic generating function for a linear symplectic twist map $L_k(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + A_k^{-1}\boldsymbol{p}, \boldsymbol{p})$. Thus, if $L = L_N \circ \ldots \circ L_1$, condition (27.2) implies that

$$L(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + A\boldsymbol{p}, \boldsymbol{p}) \quad \text{with} \quad A = \sum_{k=1}^{dN} A_k^{-1} \tag{27.6}$$

and L is a symplectic twist map. Hence Condition (27.2) can be expressed as saying that F is asymptotically linear (and asymptotically completely integrable), in that it is close to L at ∞: (27.2) (c') shows that

$$\lim_{\|\boldsymbol{p}\| \to \infty} \frac{\|F(\boldsymbol{q}, \boldsymbol{p}) - L(\boldsymbol{q}, \boldsymbol{p})\|}{\|\boldsymbol{p}\|} = 0.$$

We leave it to the reader to show that the generating function and map conditions in (27.2) are indeed equivalent.

Example 27.3 The generalized standard map satisfies both conditions (27.4) and (27.2) .

D. History

The results of Theorem 27.1 have a rich history, which can be traced back to the Poincaré-Birkhoff fixed point theorem. Birkhoff & Lewis (1933) gave a proof of existence of infinitely many periodic orbits around an elliptic fixed point of a symplectic map - which can basically be reduced to a symplectic twist map, as we saw in Section 91. An elegant proof of the Birkhoff-Lewis Theorem given by Moser (1977) follows the lines of the sketch of the Poincaré-Birkhoff Theorem given in the introduction: since close to the elliptic periodic orbit the map is, in polar coordinates, close to integrable, the set Γ of points that only move radially under the map $F^d(\cdot) - (\boldsymbol{m}, 0)$ is a graph over the angular coordinates, for suitable choices of $\boldsymbol{m}, d$. The intersection of Γ with its image $F^d(\Gamma) - (\boldsymbol{m}, 0)$ is a set of periodic orbits. This intersection is not empty, and can be obtained by finding critical points of an appropriate function related to the generating function of the exact symplectic map. This can be repeated over infinitely many $\boldsymbol{m}, d$, where $d \to \infty$.

One of the main purposes of this book is to introduce the reader to the relatively simple variational methods that are adapted to systems (discrete or continuous, see Chapter 8) not necessarily close to integrable. Some methods were introduced for Hamiltonian systems in the seminal article of Conley & Zehnder (1983), in which they prove the existence of $n+1$ homotopically trivial ($i.e.$ $\boldsymbol{m} = 0$) periodic orbits for Hamiltonian systems on $\mathbb{T}^n \times \mathbb{R}^n$ which are linear outside of a compact set. Theorem 27.1 generalizes Conley and Zehnder's global version of the Birkhoff–Lewis Theorem (see Theorems 42.3 and 42.2 for its Hamiltonian corollaries) in two ways: in its linearly $asymptotic$ condition and in the variety of the rotation vectors. Theorem 27.1 appeared in several pieces: Kook & Meiss (1989) gave the proof of existence in the convex case. Their proof of multiplicity was corrected in Golé (1994), inspired by the work of Bernstein & Katok (1987), who also consider the close-to-integrable case. The asymptotically linear case is one of the main results of the author's thesis (see also Golé (1991)). Note that Josellis (1994) proves, at a considerable technical cost of analysis and topology, a slightly stronger theorem for Hamiltonian systems (he requires less smoothness on the system).

28. Finite Dimensional Variational Setting

Let $F = F_N \circ \ldots \circ F_1$ where each F_k is the lift of a symplectic twist map of $T^*\mathbb{T}^n$ with generating function S_k. The critical action principle in Chapter 4 tells us that finding orbits of F can be done by finding solutions of:

$$\partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) + \partial_2 S_{k-1}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k) = 0,\ k \in \mathbb{Z}. \tag{28.1}$$

The appropriate space of sequences in which to look for solutions of (28.1) corresponding to $\boldsymbol{m}, d$–points of F is:

$$\overline{\overline{\mathbf{X}}} \stackrel{\text{def}}{=} \{\overline{\boldsymbol{q}} \in (\mathbb{R}^n)^{\mathbb{Z}} \mid \boldsymbol{q}_{k+dN} = \boldsymbol{q}_k + \boldsymbol{m}\}$$

which is isomorphic to $(\mathbb{R}^n)^{dN}$: the terms $(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{dN})$ determine a whole sequence in $\overline{\overline{\mathbf{X}}}$, and we will use them as a coordinate system for this space. Finding a sequence satisfying (28.1) in $\overline{\overline{\mathbf{X}}}$, is equivalent to finding $\overline{\boldsymbol{q}} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{dN})$ which is a critical point for the periodic action function:

$$W(\overline{\boldsymbol{q}}) = \sum_{k=1}^{dN-1} S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) + S_{dN}(\boldsymbol{q}_{dN}, \boldsymbol{q}_1 + \boldsymbol{m}).$$

In fact, the proof of the critical action principle (see Proposition 23.2 and also Corollary 5.5) reduces in this case to the suggestive formula:

$$dW(\overline{\boldsymbol{q}}) = \sum_{k=1}^{dN} (\boldsymbol{P}_{k-1} - \boldsymbol{p}_k) d\boldsymbol{q}_k. \tag{28.2}$$

Similarly to the 2 dimensional case in Chapter 3, we will search for critical points of W by studying the gradient flow solution of

$$\frac{d\overline{\boldsymbol{q}}(t)}{dt} = -\nabla W(\overline{\boldsymbol{q}}(t))$$

where t is an artificial time variable. Written in components, this equation is a system of dN differential equations:

$$\dot{\boldsymbol{q}}_k = -\partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) - \partial_2 S_{k-1}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k)$$

which, for C^2 functions S_k's, defines a local flow φ^t on $\overline{\overline{\mathbf{X}}}$. This flow is defined for all $t \in \mathbb{R}$ whenever the second derivatives of the S_k's are bounded (as is the case in the Standard Map): the vector field $-\nabla W$ is then globally Lipschitz (see Lang (1983)). But the existence of a local flow, garanteed by the existence of the second derivatives is enough for our purpose.

We need to complicate matters some more, to take advantage of the topology induced by the periodicity of the generating functions. Formally, this can be done by remarking that W is invariant under the diagonal $\mathbb{Z}^n$ action: $W \circ \tau_{\boldsymbol{n}} = W,\ \boldsymbol{n} \in \mathbb{Z}^n$ where

$$\tau_{\boldsymbol{n}}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{dN}) = (\boldsymbol{q}_1 + \boldsymbol{n}, \ldots, \boldsymbol{q}_{dN} + \boldsymbol{n}).$$

Hence W induces a function on the quotient $\overline{\mathbf{X}} \stackrel{\text{def}}{=} \overline{\overline{\mathbf{X}}}/\mathbb{Z}^n$. But we go one step further. We are not satisfied with finding distinct $\boldsymbol{m}, d$–points, but we want to make sure that different critical points of our function W correspond to different $\boldsymbol{m}, d$–$orbits$ of F. To this effect, we note that W is also invariant under the N^{th} iterate σ^N of the shift map:

$$(\sigma\overline{\boldsymbol{q}})_k = \boldsymbol{q}_{k+1}.$$

This is because $S_{k+N} = S_k$, and thus σ^N permutes circularly the terms of W. Hence we can define W successively on the quotients:

$$\overline{\mathbf{X}} = \overline{\overline{\mathbf{X}}}/\tau = \overline{\overline{\mathbf{X}}}/\mathbb{Z}^n \quad \text{and}$$
$$\mathbf{X} \stackrel{\text{def}}{=} \overline{\mathbf{X}}/\sigma^N$$

of $\overline{\overline{\mathbf{X}}}$ by the actions of $\tau_{\boldsymbol{n}},\ \boldsymbol{n} \in \mathbb{Z}^n$ and σ^N. Since the action of σ^N on critical sequences corresponds to the action of F on points of $T^*\mathbb{T}^n$, distinct critical points of W on $\mathbf{X}$ correspond to distinct orbits of F. The following lemma, due to Bernstein & Katok (1987), describes the topology of the different spaces:

Lemma 28.1 *The quotient maps:* $\overline{\overline{\mathbf{X}}} \to \overline{\mathbf{X}}$ *and* $\overline{\mathbf{X}} \to \mathbf{X}$ *are covering maps , and thus so is* $\overline{\overline{\mathbf{X}}} \to \mathbf{X}$*. The space* $\overline{\mathbf{X}}$ *is homeomorphic to* $\mathbb{T}^n \times (\mathbb{R}^n)^{dN-1}$*, whereas* $\mathbf{X}$ *is a (not always trivial) fiber bundle with base* $\mathbb{T}^n$ *and fiber* $(\mathbb{R}^n)^{dN-1}$*.*

Proof. We make the change of variables:

$$\boldsymbol{q} = \frac{1}{dN}\sum_{1}^{dN} \boldsymbol{q}_k$$
$$\boldsymbol{v}_k = \boldsymbol{q}_{k+1} - \boldsymbol{q}_k - \boldsymbol{m}/dN, \quad k \in \{1, \ldots, dN-1\}$$

and think of $\boldsymbol{q}$ as the base coordinate and $\boldsymbol{v}$ as the fiber. In these coordinates:

$$\tau_{\boldsymbol{n}}(\boldsymbol{q}, \boldsymbol{v}) = (\boldsymbol{q} + \boldsymbol{n}, \boldsymbol{v})$$
$$\sigma(\boldsymbol{q}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{dN-1}) = \left(\boldsymbol{q} + \frac{\boldsymbol{m}}{dN}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{dN-1}, -\sum_{j=1}^{dN-1} \boldsymbol{v}_j\right)$$
$$\sigma^{dN}(\boldsymbol{q}, \boldsymbol{v}) = (\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{v})$$

(the reader should verify this...) From the first equality, we get:

$$\overline{\mathbf{X}} = \overline{\overline{\mathbf{X}}}/\mathbb{Z}^n \simeq \mathbb{T}^n \times (\mathbb{R}^n)^{dN-1}.$$

The map σ^N induces a d–periodic, fixed point free diffeomorphism on $\overline{\mathbf{X}}$, and thus taking the quotient of $\overline{\mathbf{X}}$ by σ^N gives again a covering map. Finally, these coordinates show that $\mathbf{X} = \overline{\mathbf{X}}/\sigma^N$ is a fiber bundle over $(\mathbb{R}^n/\mathbb{Z}^n)/\frac{\boldsymbol{m}}{d}\mathbb{Z} \simeq \mathbb{T}^n$. □

29. Second Variation and Nondegenerate Periodic Orbits

In this section, we show a relationship between the second derivative of W and the spectrum of the differential of the map along a periodic orbit which will help us detect nondegenerate orbits variationally. We will delve more on this relationship in Section 33.

Second Variation vs. Dynamical Type. Suppose $F = F_N \circ \ldots \circ F_1$ where each F_k is a symplectic twist map and let W be defined as before.

Proposition 29.1 *The eigenvalues of the differential $DF^d_{\boldsymbol{z}}$ at a $\boldsymbol{m}, d$ periodic point $\boldsymbol{z}$ are in 1 to 1 correspondence with the the values $\lambda \in \mathbb{R}$ such that the following matrix $M(\lambda)$ is degenerate:*

$$M(\lambda) = \begin{pmatrix} S^{dN}_{22} + S^1_{11} & S^1_{12} & 0 & \ldots & 0 & \frac{1}{\lambda} S^{dN}_{21} \\ S^1_{21} & S^1_{22} + S^2_{11} & S^2_{12} & \ddots & & 0 \\ 0 & S^2_{12} & & & & \vdots \\ \vdots & \ddots & & & & 0 \\ 0 & \ldots & 0 & & & S^{dN-1}_{12} \\ \lambda S^{dN}_{12} & 0 & \ldots & 0 & S^{dN-1}_{21} & S^{dN-1}_{22} + S^{dN}_{11} \end{pmatrix}$$

where each entries represents an $n \times n$ matrix, which we have abbreviated:

$$S^k_{ij} = \partial_{ij} S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}).$$

Proof. Suppose that $(\boldsymbol{q}_1, \boldsymbol{p}_1) = \boldsymbol{z}$ is an $\boldsymbol{m}, d$ point for F. We want to solve the equation:

$$DF^d_{\boldsymbol{z}}(\boldsymbol{v}) = \lambda \boldsymbol{v} \tag{29.1}$$

with $\boldsymbol{v} \in T(T^*\mathbb{T}^n)_{\boldsymbol{z}}$. We follow MacKay & Meiss (1983): If $\overline{\boldsymbol{q}}$ corresponds to the orbit of $\boldsymbol{z}$ under the the successive F_k's, it must satisfy:

$$\frac{\partial W(\overline{\boldsymbol{q}})}{\partial \boldsymbol{q}_k} = \partial_2 S_{k-1}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k) + \partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) = 0.$$

Therefore, a "tangent orbit" $\delta\overline{\boldsymbol{q}}$ must satisfy:

$$S_{21}^{k-1}\delta \boldsymbol{q}_{k-1} + (S_{11}^{k} + S_{22}^{k-1})\delta \boldsymbol{q}_k + S_{12}^{k}\delta \boldsymbol{q}_{k+1} = 0 \tag{29.2}$$

When $\overline{\boldsymbol{q}}$ corresponds to a periodic point $(\boldsymbol{q}_1, \boldsymbol{p}_1)$, Equation (29.1) translates, in terms of the $\delta\overline{\boldsymbol{q}}$, to:

$$\delta \boldsymbol{q}_{dN+1} = \lambda \delta \boldsymbol{q}_1 \tag{29.3}$$

Clearly, equations (29.2) , (29.3) can be put in matrix form as $M(\lambda)\delta\overline{\boldsymbol{q}} = 0$ where $M(\lambda)$ is the matrix advertised in the proposition, which finishes the proof. □

Remark 29.2 This rather physical argument can be given a more mathematical footing. Consider the following:

$$\begin{aligned} T^*\mathbb{R}^n &\cong \big\{((\boldsymbol{q}_1, \boldsymbol{p}_1), \ldots, (\boldsymbol{q}_{dN}, \boldsymbol{p}_{dN}) \in (T^*\mathbb{R}^n)^{dN} \mid \; F_k(\boldsymbol{q}_k, \boldsymbol{p}_k) = (\boldsymbol{q}_{k+1}, \boldsymbol{p}_{k+1})\big\} \\ &\cong \{\overline{\boldsymbol{q}} \in (\mathbb{R}^n)^{dN+1} \mid \nabla W(\overline{\boldsymbol{q}})_k = 0, k = 1, \ldots, dN-1\} \end{aligned}$$

The first homeomorphism is between points in the space and their orbit segments of a given length, the second is given by the correspondence between orbit segments and critical points of the action. If one expresses a parameterization of an element of $T(T^*\mathbb{R}^n)$ with the first representation, one gets the orbit of a tangent vector under the differentials of the F_k's. If one uses the second identification , one gets (29.2) .

Nondegenerate Periodic Orbits. As an immediate consequence of Proposition 29.1, we have the second variation criterium for degenerate periodic orbits:

Definition 29.3 A periodic point $\boldsymbol{z}$ of period d for a symplectic twist map F is called *nondegenerate* if $DF_{\boldsymbol{z}}^d$ has no eigenvalue 1. A periodic orbit is called *nondegenerate* if one, and therefore all of its points are nondegenerate.

Lemma 29.4 *An $\boldsymbol{m}, d$ periodic point is nondegenerate for F if and only if the critical point of W to which it corresponds is nondegenerate.*

Lemma 29.2 proves in particular that the condition “all $\boldsymbol{m}, d$ orbits are nondegenerate” is equivalent to “W is a Morse function” ($i.e.$ a function all of whose critical points are nondegenerate, see Appendix 2). The following proposition shows that both properties are true for generic symplectic twist maps.

Proposition 29.5 *For generic symplectic twist maps , all periodic orbits are nondegenerate and hence all the periodic action functions W are simultaneously Morse.*

Proof. We remind the reader that a property is *generic* on a topological space if it satisfied on a *residual* set of that space, *i.e.* a countable intersection of open and dense sets. Robinson (1970), in his theorem 1Bi, proves that the set of C^k symplectic maps with nondegenerate periodic points is residual in the space of all C^k symplectic maps. He proceeds by induction on the period d of the points[11]. We want to adapt his proof to the space of C^1 symplectic twist maps . First note that, since the twist condition is open, this space is an open set in the space of C^1 exact symplectic maps. The only thing that we have to check, therefore, is that the perturbations that Robinson uses to remove degeneracy transform exact symplectic maps into exact symplectic maps. But this is not hard to check: each of these perturbations is given by composing the original map f with the time one map of the hamiltonian flow associated to a bump function in a small neighborhood of a given periodic point. Hence the perturbed map is the composition of the original exact symplectic map with the time 1 map of a Hamiltonian, also exact symplectic by Theorem 59.7. The composition of two exact symplectic maps being exact symplectic, we are done. □

30. The Coercive Case

The standing assumption in this section is that $F = F_N \circ \ldots \circ F_1$ where F_k is a symplectic twist map with generating function S_k satisfying the coercion condition:

Corollary 30.1 *Let F satisfy the coercion condition (27.1) . Then, for each relatively prime, $\boldsymbol{m}, d$, there is a minimum for the corresponding periodic action function W and hence an $\boldsymbol{m}, d$–point for F.*

[11] C.Robinson actually deals with higher order resonances as well, i.e, roots of unity in the spectrum of $Df_{\boldsymbol{z}}^d$.

Proof . Identical to that of Proposition 11.1. □

We have thus found at least one $\boldsymbol{m}, d$–orbit corresponding to a minimum of W. The reader should be aware that, unlike the 1 degree of freedom case, this does not imply that the orbit is a global minimizer (see Herman (1990) and Arnaud (1989)). We now turn to the multiplicity of orbits. Remember that $\mathbf{X}$ is a bundle over $\mathbb{T}^n$. Let $\Sigma \cong \mathbb{T}^n$ be its zero section. Let $K > \sup_{\overline{\boldsymbol{q}} \in \Sigma} W(\overline{\boldsymbol{q}})$. Trivially, we have:

$$\Sigma \subset W^K \stackrel{\text{def}}{=} \{\overline{\boldsymbol{q}} \in \mathbf{X} \mid W \leq K\}$$

Note that since W is proper, for almost every K, W^K is a compact manifold with boundary, by Sard's Theorem. From this we get the commutative diagram in homology:

$$\begin{array}{ccc} H_*(\Sigma) & \xrightarrow{\;k_*\;} & H_*(\mathbf{X}) \\ i_* \searrow & & \nearrow j_* \\ & H_*(W^K) & \end{array} \tag{30.1}$$

where i, j, k are all inclusion maps. But $k_* = Id$ since Σ and $\mathbf{X}$ have the same homotopy type. Hence i_* must be injective. If all the $\boldsymbol{m}, d$–points are nondegenerate, W is a Morse function (a generic situation by Proposition 29.5) and according to Morse Theory (see Section 61 in Appendix 2) W^K has the homotopy type of a finite CW complex, with one cell of dimension k for each critical point of index k in W^K. In particular, we have the following Morse inequalities:

$$\#\{\text{critical points of index } k\} \geq b_k$$

where b_k is the kth Betti number of W^K. In the present case $b_k \geq \binom{n}{k}$ since $H_*(\mathbb{T}^n) \hookrightarrow H_*(W^K)$. Hence there are at least $\sum_{k=1}^{n} \binom{n}{k} = 2^n$ critical points in this nondegenerate case. If W is not a Morse function, rewrite the diagram (30.1), but in cohomology, reversing the arrows and raising the stars. Since $k^* = Id$, j^* must be injective this time. We know that the cup length $cl(X) = cl(\mathbb{T}^n) = n + 1$. By definition, this means that there are n cohomology classes $\alpha_1, \ldots, \alpha_n$ in $H^1(\mathbf{X})$ such that $\alpha_1 \cup \ldots \cup \alpha_n \neq 0$. Since j^* is injective, $j^*\alpha_1 \cup \ldots \cup j^*\alpha_n \neq 0$ and thus $cl(W^K) \geq n + 1$. W^K being compact, and strictly forward invariant under the gradient flow, we can apply Lyusternik-Schnirelman theory which implies that W has at least $n+1$ critical points in W^K (The proof of Theorem 1 in CH.2 Section 19 of Dubrovin & al. (1987) , which is for compact manifolds without boundaries can easily be adapted to this case.) □

31. Asymptotically Linear Systems

In this section we swap the coercion condition (27.1) for asymptotic linearity of the map (27.2). In this case, the periodic action function W does not necessarily have any minimum. The topological tool we use here is Proposition 64.6. We roughly sketch the content and philosophy of this proposition in the next section.

We remind the reader of our assumption (27.2) : $F = F_N \circ \ldots \circ F_1$ is a product of lifts of symplectic twist maps of $T^*\mathbb{T}^n$. The generating function S_k of F_k satisfies:

$$S_k(\boldsymbol{q}, \boldsymbol{Q}) = \frac{1}{2}\langle A_k(\boldsymbol{Q} - \boldsymbol{q}), (\boldsymbol{Q} - \boldsymbol{q})\rangle + R_k(\boldsymbol{q}, \boldsymbol{Q})$$

with:

$$(27.2) \qquad A_k = A_k^t,\ \det\ A_k \neq 0, \quad \det\ \sum_1^N A_k^{-1} \neq 0, \quad \lim_{\|\boldsymbol{Q}-\boldsymbol{q}\|\to\infty} \frac{\nabla R_k(\boldsymbol{q}, \boldsymbol{Q})}{\|\boldsymbol{Q} - \boldsymbol{q}\|} = 0$$

We view R_k as a *global* perturbation term. As before we let $L_k(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + A_k^{-1}\boldsymbol{p}, \boldsymbol{p})$ and $L = L_N \circ \ldots \circ L_1$. Then $L(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + A\boldsymbol{p}, \boldsymbol{p})$ with $A = \sum_1^N A_k^{-1}$. It is crucial to note that, because of the conditions on the determinants of A_k and $\sum A_k^{-1}$, L and all the L_k's are completely integrable symplectic twist maps .

As before, we are looking for critical points of:

$$W(\overline{\boldsymbol{q}}) = \sum_{k=1}^{dN} S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) = \sum_{k=1}^{dN} \frac{1}{2}\left\langle A_k(\boldsymbol{q}_{k+1} - \boldsymbol{q}_k), (\boldsymbol{q}_{k+1} - \boldsymbol{q}_k)\right\rangle + \sum_{k=1}^{dN} R_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}).$$

where $\overline{\boldsymbol{q}} \in \overline{\overline{\overline{\mathbf{X}}}}$ i.e., $\boldsymbol{q}_{dN+1} = \boldsymbol{q}_1 + \boldsymbol{m}$. The first sum in the right hand side is the (quadratic) action function for the integrable symplectic twist map L defined above. We change coordinates $\Psi : (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{dN-1}) \mapsto (\boldsymbol{q}, \boldsymbol{v})$ as in Section 28:

$$\boldsymbol{q} = \frac{1}{dN}\sum_1^{dN} \boldsymbol{q}_k$$
$$\boldsymbol{v}_k = \boldsymbol{q}_{k+1} - \boldsymbol{q}_k - \boldsymbol{m}/dN, \quad k \in \{1, \ldots, dN - 1\}.$$

In these coordinates, W is of the form:

$$W(\boldsymbol{q}, \boldsymbol{v}) = \mathcal{Q}(\boldsymbol{v}) + \boldsymbol{a} \cdot \boldsymbol{v} + \boldsymbol{b} + R(\boldsymbol{q}, \boldsymbol{v})$$

where $R = \sum_1^{dN} R_k \circ \Psi^{-1}$, $\mathcal{Q}$ is a homogeneous quadratic function and $\boldsymbol{a}, \boldsymbol{b}$ are constant vectors. The function $W_L(\boldsymbol{v}) \stackrel{def}{=} \mathcal{Q}(\boldsymbol{v}) + \boldsymbol{a}\cdot\boldsymbol{v} + \boldsymbol{b}$ comes up while replacing $\boldsymbol{q}_{k+1} - \boldsymbol{q}_k$ by $\boldsymbol{v}_k + \boldsymbol{m}/dN$ in the quadratic part of W. Thus W_L is the action function of the integrable map L in our new coordinates.

Postponing the proof that $\mathcal{Q}(v)$ is nondegenerate, we conclude the proof of the theorem. The maps $\tau_{\boldsymbol{n}}$ $(\boldsymbol{n} \in \mathbb{Z}^n)$, and σ introduced in Section 28 all map fiber to fiber diffeomorphically and linearly in the trivial bundle $\overline{\overline{\overline{\mathbf{X}}}} \to \mathbb{R}^n$ with projection $(\boldsymbol{q}, \boldsymbol{v}) \mapsto \boldsymbol{q}$. Hence $\mathcal{Q}(\boldsymbol{q}, \boldsymbol{v}) = \mathcal{Q}(\boldsymbol{v})$, which is quadratic nondegenerate in the fibers, induces in the quotient $\mathbf{X}$ of $\overline{\overline{\overline{\mathbf{X}}}}$ a function $\mathcal{Q}$ which is also quadratic nondegenerate in the fibers of the bundle $\mathbf{X} \to \mathbb{T}^n$. Finally, it is easy to see that the asymptotic condition on R_k given in (27.2) implies that:

$$\frac{1}{\|\boldsymbol{v}\|}\frac{\partial}{\partial \boldsymbol{v}}(W - \mathcal{Q}) = \frac{1}{\|\boldsymbol{v}\|}\left(\frac{\partial R}{\partial \boldsymbol{v}} + \boldsymbol{a}\right) \to 0 \quad \text{as} \quad \|\boldsymbol{v}\| \to \infty$$

in $\overline{\overline{\overline{\mathbf{X}}}}$ and hence also in its quotient $\mathbf{X}$. Hence W is a gpqi, in the sense of Proposition 64.6 which provides the estimates advertised and concludes the proof of Theorem 27.1.

We now turn to the proof that, given the assumption (27.2) , $\mathcal{Q}(\boldsymbol{v})$ is nondegenerate. The reader could work the linear algebra out directly. We prefer to give a dynamical argument which might enlighten us a bit about the linear asymptotic condition. We need to show that the quadratic form $\mathcal{Q}$ has zero kernel. This is true if and only if the linear equation $d\mathcal{Q}(\boldsymbol{v}) + \boldsymbol{a} = 0$ has a unique solution. Now, $\boldsymbol{m}, d$ orbits of L are in one to one correspondence with critical points of W_L in $\overline{\overline{\overline{\mathbf{X}}}}$, $i.e.$ solutions of the following linear equations:

$$\begin{aligned} &\frac{\partial W_L}{\partial \boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{v}) = 0 \\ &d\mathcal{Q}(\boldsymbol{v}) + \boldsymbol{a} = 0. \end{aligned} \tag{31.1}$$

Since W_L does not depend on $\boldsymbol{q}$, the first equation is always satisfied. Since the second equation does not depend on $\boldsymbol{q}$ either, one solution of this equation yields exactly an n-dimensional plane of solutions of the form $(\boldsymbol{q}, \boldsymbol{v}^*)$, for a fixed $\boldsymbol{v}^*$. We now show dynamically that the set of solution of Equations (31.1) is indeed n dimensional, thereby proving that the second equation has a unique solution. The solutions of (31.1) in $\overline{\overline{\overline{\mathbf{X}}}}$ are in one to one correspondence with the $\boldsymbol{m}, d$ points of the map L. Since L is a linear, completely integrable symplectic twist map , these orbits form an n dimensional plane parallel to the 0 section of

$T^*\mathbb{T}^n$. Since the generating function of L is quadratic and the above change of coordinate Ψ is affine, this plane corresponds 1-1 to an n-plane of critical points of $W_L(\boldsymbol{q}, \boldsymbol{v})$ in $\overline{\overline{\mathbf{X}}}$. □

32. Ghost Tori

On the Topological Part of the Proof of Theorem 27.1. In the proof of Theorem 27.1, we invoked a topological "black box", Proposition 64.6. This proposition says that, if a function W is asymptotically quadratic (in a precise sense) on a fiber bundle over a compact manifold M (the bundle is $\mathbf{X}$ and the manifold is $M = \mathbb{T}^n$ here), then the number of critical point of the function W is regimented by the cohomology of M. We now try in a paragraph to peal the successive layers that constitute the proof of Proposition 64.6 in order to extract the gist of the ideas, and motivate the concept of ghost torus. Please see Appendix 2 for the rigorous definitions of the concepts and for the proofs of the following statements.

The final layer in the proof of Proposition 64.6 (in the proof of the proposition itself and in the proof of Proposition 64.1) consists, through changes of variables, in coming back to the simpler situation of Proposition 62.4. This latter proposition investigates the gradient flow of a function which has an isolating block B of the form:

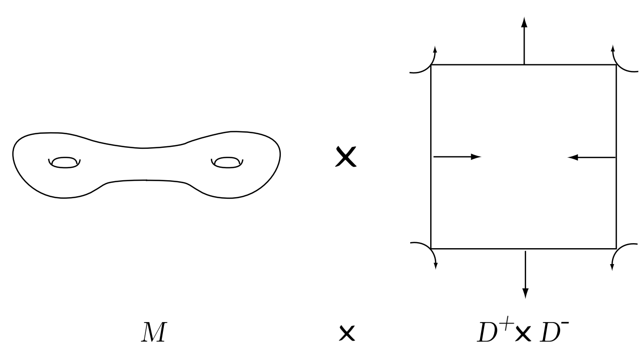

Fig. 32.0. The isolating neighborhood in Proposition 62.4.

Here, D^+, D^- are homeomorphic to disks of some euclidean spaces. The picture suggests that the flow leaves the boundary of the block in either positive or negative time. The proof of Proposition 62.4 uncovers (via Lemma 63.4) a relationship between the topology of the largest invariant set [(12)] G for the gradient flow in B and the manifold M. More

[12]This set is often denoted by I instead of G in Appendix 2.

precisely, one proves the existence of an injection $H^*(M) \hookrightarrow H^*(G)$. This a way to say that the set G is at least as topologically complex as the manifold M.

Ghost Tori. As roughly explained in the previous paragraph, the topological part of the proof of Theorem 27.1 brings about the existence of an invariant set G for the gradient flow of the action function W on $\mathbf{X}$. We call this set G a *ghost torus,* because it lives in the nether world of sequences (vs. the "real" world $T^*\mathbb{T}^n$ where the dynamics takes place) and has a topology at least as complex as that of the torus: $H^*(\mathbb{T}^n) \hookrightarrow H^*(G)$. Looking a little more closely at the proof of Proposition 64.6, one would see that G is in fact made of all the bounded orbits of the gradient flow of W in $\mathbf{X}$. As we did with ghost circles in Chapter 3, we can indeed think of the ghost tori as the ghosts of the tori of $T^*\mathbb{T}^n$ invariant under a completely integrable symplectic twist map L. Indeed, the L-invariant torus of rotation vector $\boldsymbol{m}/d$ has a homeomorphic counterpart in the set $\mathbf{X}$ of sequences, namely, the critical sequences corresponding to the periodic this torus is filled with. This torus of critical point is the ghost torus of rotation vector $\boldsymbol{m}/d$ for L: it is easy to see that the gradient flow has no other bounded orbits.

What are the ghost tori good for? My hope when I introduced the concept in Golé (1989) was to prove the existence of $irrational$ ghost tori, and thus a generalization of the Aubry-Mather Theorem. Indeed, the only ghost tori whose existence we can secure for all symplectic twist maps (at this point, and to my knowledge) are those in periodic, $\boldsymbol{m}, d$-sequence spaces. This prompted my investigation of the dimension 2 case in Golé (1992 a), where the existence of ghost circles of all rotation number was indeed proven. Ghost circles turned out to be very useful in ordering Aubry-Mather sets as well, see Chapter 3. [Beware that the notion of ghost circle is actually more restrictive than that of ghost torus: a ghost circle may miss some bounded orbits for the gradient flow in a given $\mathbf{X}$ and be a proper subset of the ghost torus of a particular rotation number].

Unfortunately, two of the main tools that make things work in the dimension 2 case are the monotonicity of the flow with respect to the natural order on numerical sequences, and the notion of cyclic order. These two notions break down in higher dimension, and my attempt to find irrational ghost tori by other means ($eg.$ proving some regularity of subsets of ghost tori and taking limits, or using Conley continuation in appropriate Banach spaces) have been unsuccessful. At this point, I am rather pessimistic that such a program could be carried out.

On the other hand, the construction and point of view ghost tori are based on are, for those who know it, very reminiscent of that of Floer Cohomology, where the set of loops of bounded action is used to construct the Floer Cohomology complex. Indeed, the variational/topological theory involved in this chapter and in Chapter 8 could be interpreted as a discrete version of Floer's theory for cotangent bundles [(13)]. I hope it could be put to some of the use Floer's Theory has. It is interesting to note, for instance, that Floer's theory has concentrated on homotopically trivial periodic orbits – which is not the case here.

33. Hyperbolicity vs. Action Minimizers

Dynamical Types of Periodic Orbits. We now return to the connection between the dynamical type of a periodic orbit and the spectrum of Hessian of the action along that orbit, which we started investigating in Section 29. For more detail, the reader is urged to consult MacKay & Meiss (1983) and Kook & Meiss (1989).

In Section 56 of Appendix 1, it is shown that the dynamics of a linear symplectic map around the origin in $\mathbb{R}^{2n}$, which is a fixed point for such a map, are determined by the spectrum of the map. This spectrum has the special property that if λ is an eigenvalue, than so are $1/\lambda, \overline{\lambda}$ and $1/\overline{\lambda}$. As a consequence, $\mathbb{R}^{2n}$ can be decomposed in even dimensional eigenspaces of 3 flavors: elliptic (corresponding to pairs of conjugate eigenvalues on the unit circle), parabolic (double eigenvalues 1 or -1), and hyperbolic (either real pairs $\lambda, 1/\lambda$ or complex quadruplets $(\lambda, 1/\lambda, \overline{\lambda}, 1/\overline{\lambda})$). The origin is stable only if it is an elliptic fixed point (or parabolic, if there are as many eigenvectors as the multiplicity of the eigenvalue ± 1). We can now apply this stability analysis to periodic orbits of symplectic twist maps, by considering the spectrum of DF^N, where N is the period. There is obviously a loss of control when taking this step. Hyperbolicity of the differential at a periodic orbit implies that of the map itself (by the Hartmann-Grobman theorem, see Robinson (1994)), so instability persists when passing from linear to nonlinear. On the other hand stability does not necessarily survive. Indeed, in the elliptic case, stability cannot be guaranteed while perturbing a linear simplectic map, except in dimension 2, when a certain, generic, twist condition is satisified (Moser (1973) , Theorem 2.11). This is because KAM theory (see Chapter 6),

[13] Floer's theory originally took place on compact manifolds. See Cielieback (1992) for a cotangent bundle version of Floer's theory.

which guarantees stability in dimension 2, does not anymore in higher dimensions (an n dimensional torus does not separate $T^*\mathbb{T}^n$, unless $n = 2$. However, these invariant tori can be "sticky" and provide long term stability. At any rate, knowing the linear type of the orbit gives a good amount of information about the dynamics around this orbit.

Hyperbolicity vs. Minimization: The Periodic Case. If $\lambda_1, 1/\lambda_1, \dots, \lambda_n, 1/\lambda_n$ are the eigenvalues of a linear symplectic map T, a convenient way to track down stability is with the *traces* $\rho_k = \lambda_k + 1/\lambda_k$ and *residues* $R_k = \frac{1}{4}(2 - \rho_k)$. Indeed, it is not hard to see that the origin is stable only when the traces are real and fall in $[-2, 2]$, or the residues are real and in $[0, 1]$. Moreover the characteristic polynomial can be neatly expressed in terms of the traces (use $1/\lambda(\lambda_k - \lambda)(1/\lambda_k - \lambda) = \rho - \rho_k$):

$$\det\ (T - \lambda Id) = \lambda^n \prod_{k=1}^{n} (\rho - \rho_k). \tag{33.1}$$

We now look back at the matrix $M(\lambda)$ introduced in Section 29:

$$M(\lambda) = \begin{pmatrix} S_{22}^{dN} + S_{11}^1 & S_{12}^1 & 0 & \dots & 0 & \frac{1}{\lambda} S_{21}^{dN} \\ S_{21}^1 & S_{22}^1 + S_{11}^2 & S_{12}^2 & \ddots & & 0 \\ 0 & S_{12}^2 & & & & \vdots \\ \vdots & \ddots & & & & 0 \\ 0 & \dots & 0 & & & S_{12}^{dN-1} \\ \lambda S_{12}^{dN} & 0 & \dots & 0 & S_{21}^{dN-1} & S_{22}^{dN-1} + S_{11}^{dN} \end{pmatrix}.$$

We showed in Section 29 that the eigenvalues of $T = DF_{\boldsymbol{q}_*}^N$ are exactly the solutions of the equation $\det\ M(\lambda) = 0$. Hence we must have

$$\det\ M(\lambda) = C \prod_{k=1}^{n} (\rho - \rho_k) \tag{33.2}$$

for some constant C. It is not too hard to see that C is in fact the product of the determinants of (minus) the superdiagonal blocks: $C = \prod_{k=1}^{N-1} \det\ (-S_{12}^k)$, which, if we assume the convexity condition (27.4) of Lemma 27.2, or just the ordinary positive twist condition for the dimension 2 case, happens to be positive. Finally, we set $\lambda = 1$ in (33.2) . In this case, note that $M(1) = HessW(\boldsymbol{q}_*), \rho - \rho_k = 4R_k$ and we obtain:

$$\prod_{k=1}^{n} R_k = \left(\frac{1}{4}\right)^n \frac{\det\ HessW(\boldsymbol{q}_*)}{\prod_{k=1}^{N-1} \det\ (-S_{12}^k)}.$$

In the twist map case $n = 1$, there is only one residue R_1, which is negative if $HessW(\boldsymbol{q}_*)$ is positive definite. This gives the following, due to MacKay & Meiss (1983):

Theorem 33.1 (MacKay & Meiss) *Let $\boldsymbol{z}_*$ be a periodic point for a positive twist map of the cylinder, corresponding to a critical sequence $\boldsymbol{q}_*$. Then, if $\boldsymbol{q}_*$ is a nondegenerate local minimum for the periodic action, the orbit of $\boldsymbol{z}_*$ is hyperbolic.*

The Infinite Orbit Case. It is interesting to note that Theorem 33.1 has counterexamples in higher dimensions (see Arnaud (1989), where open sets of maps with no hyperbolic fixed points are found arbitrarily close to an integrable map, even though the action does have a minimum). However there is a higher dimensional result, due to Aubry et. al. (1991) which relates hyperbolicity to a *stronger* form of minimization of arbitrary orbits. Given a stationary infinite configuration $\boldsymbol{q}$ for a symplectic twist map on $T^*\mathbb{T}^n$ (*i.e.* the $\boldsymbol{q}$ coordinate of a (*not necessarily periodic*) orbit of the map), the Hessian of W as the following *infinite* matrix, similar to $M(1)$:

$$HessW(\boldsymbol{q}) = \begin{pmatrix} \ddots & \ddots & & & & & \\ \beta_0 & \alpha_1 & \beta_1 & & & & \\ & \beta_1 & \alpha_2 & \beta_2 & & & \\ & & \beta_2 & \alpha_3 & \ddots & & \\ & & & \ddots & \ddots & \beta_{q-1} & \\ & & & & \beta_{q-1} & \alpha_q & \beta_q \\ & & & & & \ddots & \ddots \end{pmatrix}$$

with:

$$\alpha_k = \partial_{22}S(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k) + \partial_{11}S(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}), \quad \beta_k = \partial_{12}S(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k).$$

If $\boldsymbol{q}$ is a local minimizer, *i.e.* minimizes W locally on any finite segment, then the spectrum of $HessW(\boldsymbol{x})$ is positive (here $HessW$ is seen as an operator on "variation" sequences in l^2, *i.e.* those sequences $\boldsymbol{y}$ such that $\sum \|y_k\|^2 < \infty$). We say that $\boldsymbol{q}$ has a *phonon gap* if moreover Spec$(HessW(\boldsymbol{q})) \in [a, \infty),\ a > 0$. An invariant set has a phonon gap a if each of the orbits it contains does, and if their phonon gaps are all greater or equal to a.

Theorem 33.2 (Aubry–Baesens–MacKay) *Let Λ be a closed invariant set for a symplectic twist map of $\mathbb{R}^{2n}$ and Λ' be the associated set of critical sequences for W.*

Suppose that $\partial_{12}S(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$, $(\partial_{12}S)^{-1}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}), \partial_{11}S(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$ and $\partial_{22}S(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$ are all bounded for $\boldsymbol{q} \in \Lambda', k \in \mathbb{Z}$. Then Λ is uniformly hyperbolic if and only if Λ has a phonon gap.

Concluding Remarks. 1) We have not talked about the different types of bifurcations that govern the changes of periodic orbits from one type to another. We refer to Kook & Meiss (1989) and the references therein for more information about this vast and interesting subject. 2) The above theorems are related to a general principle, first unveiled by Morse in Riemannian geometry. In that context, Morse (see Milnor (1969)) revealed a tie between the index of the second derivative of the action of a segment of geodesic and the number of conjugate points this segment has. In terms of more general Lagrangian systems, this number can be formulated as a certain rotation index (the Maslov index) of Lagrangian subspaces under the differential of the flow along an orbit segment (see Duistermaat (1976), Conley & Zehnder (1984)). If the orbit is hyperbolic, the Lagrangian tangent subspace can be chosen to be the unstable manifold. It would be interesting to cast the above theorems in a symplectic setting, using the invariance under symplectic reduction of the Maslov index from the manifold $graph(dW)$ to that of $graph(F^N)$.