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THE AUBRY-MATHER THEOREM

8. Introduction

A. Motivation and Statement of the Theorem

The orbits of the twist map f_0 whose lift is the completely integrable shear map given by $F_0(x, y) = (x + y, y)$, possess the following four fundamental properties, some of which we have yet to define:

(1) They lie on invariant circles which are graphs over the circle $\{y = 0\}$.

(2) They are ordered cyclically, like orbits of rotations on the circle.

(3) They come with all rotation numbers in $(-\infty, +\infty)$.

(4) They are action minimizers.

The KAM theorem (see the Introduction and 34.1) implies that, in the measure sense, most of these invariant circles will "survive" a $small$ perturbation of f_0. The rotation numbers of these survivors has to be very irrational (diophantine). One cannot hope for all these circles to survive under arbitrary perturbation of the map f_0. In fact, it is known numerically that that for $k > 0.9716354$, the standard map has no invariant circle (see Meiss (1992)). In the context of the Standard family, the Aubry-Mather theorem implies that, for each invariant circle of f_0, and for each $\lambda > 0$, there exists an invariant set for f_λ which can be seen as the remnant of the invariant circle. We will define the terminology (cyclically ordered, minimizers, Denjoy sets etc...) in subsequent sections.

Theorem 8.1 (Aubry-Mather) *Let* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *be the lift of a* C^2 *twist map of the cylinder with generating function* S *satisfying the following growth or* ***coercion***[6] *condition:*

$$\lim_{|X-x|\to\infty} S(x, X) \to +\infty \tag{8.1}$$

Then F *has orbits of all rotation numbers in* $\mathbb{R}$*. Moreover, these orbits can be chosen to have the following properties:*

(1) They are cyclically ordered

(2) They lie on closed F*-invariant sets, called* ***Aubry-Mather sets*** *that form graphs over their projection on the circle* $\{y = 0\}$ *and that are conjugated to closed invariant sets of lifts of circle homeomorphisms: either lifts of periodic orbits, Denjoy Cantor sets (and optionally, orbits homoclinic to these sets) or the full real line.*

(3) They may be chosen to be action minimizers.

We will see that an invariant Cantor sets must occur each time there is no invariant circle of a given irrational rotation number. The existence of these invariant Cantor sets was the striking novelty of this theorem. For this reason, the term "Aubry-Mather sets" is sometimes restricted to denote only the invariant Cantor sets of action minimizers.

Sketch of the Proof. We will find periodic orbits of all rational rotation numbers by minimizing the periodic action W_{mn} on the space $X_{m,n}$ of m, n sequences (see Proposition 5.7 for definitions). Aubry's Fundamental Lemma will imply that W_{mn}-minimizers are "cyclically ordered", *i.e.* ordered like orbits of circle homeomorphisms. The cyclic order (CO) property enables us to take limits of these periodic orbits (they will be in a compact set of sequences if their rotation numbers are in a bounded set). Cyclic order also implies that the rotation number of the limiting orbit exists and is the limit of the rotation numbers of the periodic orbits.

One way in which this presentation differs from the excellent surveys of this subject by Meiss (1992) or Hasselblat & Katok (1995) is the focus on the cyclic order property at

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[6] This is not quite the usual use of the term coercive. Usually, a numerical function ϕ on a normed space is called *coercive* if $\lim_{\|u\|\to\infty} \phi(u) = +\infty$.

the level of sequences (that are not necessarily realized by orbits). I found it a convenient bridge between the study of the dynamics of circle homeomorphisms (which appears in the appendix to this chapter) and that of Aubry-Mather sets.

Aubry-Mather Theorem as Topological Stability. There are important notions in the theory of dynamical systems that help to compare different systems. We refer to Hasselblat & Katok (1995) for more details. Suppose $f : M \to M$ and $g : N \to N$ are two $C^r, r \geq 0$ maps on manifolds. We say that f and g are *topologically conjugate* if there is a homeomorphism $h : M \to N$ such that $h \circ f = g \circ h$. Orbits of conjugate maps are in 1-1, continuous correspondence (given by the map h). If the map h is continuous but only surjective (and not necessarily injective), we say that g is a *factor* of f and we call h a *semiconjugacy*. Finally, if f is a diffeomorphism and if it is a factor of any homeomorphisms in a C^0 neighborhood of it, we say that f is topologically stable. In light of this terminology, we can say that the Aubry Mather theorem is a "weak" stability statement: All maps in a C^1 neighborhood of the completely integrable map have the completely integrable map restricted to irrational rotation invariant circles as a factor.

B. From the Annulus to the Cylinder

We precede our study by a Lemma, which implies that we can reduce our study to twist maps of the cylinder.

Lemma 8.2 *Let f be a $C^k, k \geq 2$, twist map of a compact annulus $\mathcal{A}$. Then f can be extended to a C^k twist map of the cylinder $\mathcal{C}$, in such a way that it coincides with the shear map $(x, y) \mapsto (x + cy, y)$ outside a compact set. In particular, the generating function of any lift of the extended map satisfies the growth condition $\lim_{|X-x|\to\infty} S(x, X) \to +\infty$.*

To prove this lemma, one extends the generating function S from $\psi(A)$ to $\mathbb{R}^2$ by interpolating it to the quadratic $\frac{c}{2}(X - x)^2$ outside of some appropriate compact set. See Forni & Mather (1994) or Moser (1986a) . As a corollary of this lemma, we obtain the following version of the Aubry-Mather theorem:

Theorem 8.3 (Aubry-Mather on the compact annulus) *Let F be the lift of a twist map of the bounded annulus and suppose that the rotation numbers of the restriction of F to the lower and upper boundaries are ρ_-, and ρ_+ respectively. Then F has orbits of all rotation numbers in $[\rho_-, \rho_+]$. These orbits are minimizers, recurrent, cyclically ordered and they lie on compact invariant sets that form (uniformly) Lipschitz graphs over their projections. These sets may either be periodic orbits, invariant circles or invariant Cantor sets on which the map is semi-conjugate to lifts of circle rotations.*

9. Cyclically Ordered Sequences and Orbits

If a map $G : \mathbb{R} \to \mathbb{R}$ is the lift of a circle homeomorphism which preserves the orientation, it is necessarily strictly increasing and must satisfy $G(x+1) = G(x)+1$. Hence, if $\{x_k\}_{k\in\mathbb{Z}}$ is an orbit of G, it must satisfy:

$$x_k \leq x_j + p \Rightarrow x_{k+1} \leq x_{j+1} + p, \ \forall\, k, j, p \in \mathbb{Z}. \tag{9.1}$$

We will say that a sequence $\{x_k\}_{k\in\mathbb{Z}}$ in $\mathbb{R}^{\mathbb{Z}}$ is *Cyclically Ordered,* (or *CO* in short) if it satisfies (9.1) . Clearly the CO sequences form a closed set for the topology of *pointwise convergence* in $\mathbb{R}^{\mathbb{Z}}$: $\boldsymbol{x}^{(j)} \to \boldsymbol{x}$ whenever $x_k^j \to x_k$ for all k. Note that this topology is the same as the product topology on the space of sequences. Using the *partial order on sequences* (it comes with three degrees of strictness):

$$\boldsymbol{x} \leq \boldsymbol{y} \Leftrightarrow \{\forall k, x_k \leq y_k\}$$
$$\boldsymbol{x} < \boldsymbol{y} \Leftrightarrow \{\forall k, x_k \leq y_k \quad \text{and} \quad \boldsymbol{x} \neq \boldsymbol{y}\}$$
$$\boldsymbol{x} \prec \boldsymbol{y} \Leftrightarrow \{\forall k, x_k < y_k\}$$

we let the reader check that an equivalent definition of CO sequences is:

$$\forall m, n \in \mathbb{Z}, \qquad \tau_{m,n}\boldsymbol{x} \geq \boldsymbol{x} \quad \text{or} \quad \tau_{m,n}\boldsymbol{x} \leq \boldsymbol{x} \tag{9.2}$$

where

$$(\tau_{m,n}\boldsymbol{x})_k = x_{k+m} + n.$$

We say that the orbit $\{(x_k, y_k)\}_{k\in\mathbb{Z}}$ of a twist map is a *Cyclically Ordered orbit* or *CO orbit* if $\{x_k\}_{k\in\mathbb{Z}}$ is CO. These orbits come with various other names in the literature: *Well Ordered* (does not evoque the *cyclic* ordering), *Monotone* (is used in too many contexts), *Birkhoff* (this order was apparently never mentioned by Birkhoff).[7] The following lemma, whose proof is in great part due to Poincaré (1885), is central to our use of CO sequences.

Lemma 9.1 *Let $\{x_k\}_{k\in\mathbb{Z}}$ be a CO sequence then $\rho(\boldsymbol{x}) = \lim_{k\to\infty} x_k/k$ exists and:*

$$|x_k - x_0 - k\rho(\boldsymbol{x})| \le 1. \tag{9.3}$$

Moreover $\boldsymbol{x} \to \rho(\boldsymbol{x})$ is a continuous function on CO sequences, with the topology of pointwise convergence.

Define:

$$CO_{[a,b]} = \{\boldsymbol{x} \in CO \mid \rho(\boldsymbol{x}) \in [a,b]\}.$$

The following lemma shows that it is easy to find limits of CO sequences, as long as their rotation numbers are bounded.

Lemma 9.2 *The sets $CO_{[a,b]}/\tau_{1,0}$ and $CO_{[a,b]} \cap \{\boldsymbol{x} \in \mathbb{R}^{\mathbb{Z}} \mid x_0 \in [0,1]\}$ are compact for the topology of pointwise convergence.*

We give the proofs of both these lemmas in the appendix to this chapter. The fact, given by these lemmas, that the rotation number behaves well under limits of CO-sequences is one of the essential points in the theory of twist maps that does not generalize to higher dimensional maps: to our knowledge, there is no dynamically natural definition of CO sequences in $\mathbb{R}^n, n \ge 2$ which ensures the existence of rotation vectors which behave well under limits. Note that there is, however, a natural generalization of CO sequences in the context of maps $\mathbb{Z}^d \to \mathbb{R}$, see Chapter 9.

There is a visual way to describe CO sequences, which we now come to. A sequence $\boldsymbol{x}$ in $\mathbb{R}^{\mathbb{Z}}$ is a function $\mathbb{Z} \to \mathbb{R}$. One can interpolate this function linearly and obtain a piecewise affine function $\mathbb{R} \to \mathbb{R}$ that we denote by $t \mapsto x_t$. The graph of this function is sometimes

[7] This is not an indictment of the authors who have used these terminologies: the author of this book has himself used them all in various publications...

called the *Aubry diagram* of the sequence. We say that two sequences $\boldsymbol{x}$ and $\boldsymbol{w}$ *cross* if their corresponding Aubry diagrams cross. There are two types of crossing: at an integer k, in which case $(x_{k-1} - w_{k-1})(x_{k+1} - w_{k+1}) < 0$ or at a non integer $t \in (k, k+1)$, in which case $(x_k - w_k)(x_{k+1} - w_{k+1}) < 0$. These inequalities can be taken as a definition of crossings. Non–crossing of two sequences can be put in terms of the partial order on sequence: $\boldsymbol{x}, \boldsymbol{y}$ do not cross if and only if $\boldsymbol{x} \leq \boldsymbol{y}$. *In particular a sequence* $\boldsymbol{x}$ *is CO if and only if it has no crossing with any of its translates* $\tau_{m,n}\boldsymbol{x}$.

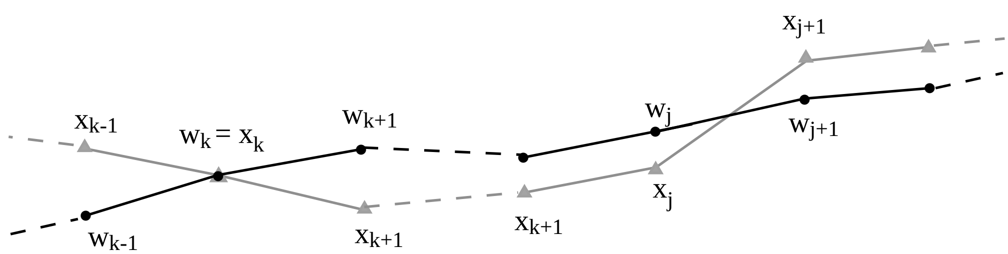

Fig. 9.0. Aubry diagrams of sequences and their crossings: in this example the sequences $\boldsymbol{x}$ and $\boldsymbol{w}$ have crossings at the integer k and between the integers j and $j+1$.

10. Minimizing Orbits

Throughout the rest of this chapter, we consider a lift F of a given twist map f of the cylinder, and its corresponding generating function S, action function W, periodic action function W_{mn} and change of variable ψ. A sequence segment $(x_k, \ldots, x_m)$ is *(action) minimizing* if

$$W(x_k, \ldots, x_m) \leq W(y_k, \ldots, y_m)$$

for any other sequence segment $(y_k, \ldots, y_m)$ with same endpoints: $x_k = y_k, x_m = y_m$. Since minimizing segments are necessarily critical for W, they correspond to orbit segments called *(action) minimizing orbit segment*. A bi-infinite sequence is called a *(global action) minimizer* if any of its segments is minimizing. The orbit it corresponds to is a *minimizing orbit*, or simply *minimizer*, when the context is clear. Note that the set of minimizers is closed under the topology of pointwise limit (see Exercise 10.5). Finally a W_{mn}*-minimizer* is a periodic sequence in $X_{m,n}$ that minimizes the function W_{mn}.

A recurrent theme in the Calculus of Variation is that minimizers have regimented crossings. In the case of geodesics on a Riemannian manifold, geodesics that (locally) minimize length cannot have conjugate points, *i.e.* small variations with fixed endpoints of a minimizing geodesic only intersect that geodesic at the endpoints (Milnor (1969)), and geodesics that minimize length globally cannot have self intersections (Manẽ (1991), page 102). We will see, in the present theory, that minimizers satisfy a non-crossing condition, which implies that W_{mn}–minimizers (and more generally, recurrent minimizers) are CO.

Lemma 10.1 (crossing) *Suppose that* $(x-w)(X-W) \leq 0$*. Then:*

$$S(x,X)+S(w,W)-S(x,W)-S(w,X) \leq 0,$$

and equality occurs iff $(x-w)(X-W)=0$

Proof. We can write:

$$S(x,X)-S(x,W)=\int_0^1 \partial_2 S(x,X_s)(X-W)ds,$$

where $X_s=(1-s)W+sX$. Applying the same process to $h(x)=S(x,X)-S(x,W)$, we get:

$$\begin{aligned} &S(x,X)+S(w,W)-S(x,W)-S(w,X)=h(x)-h(w)= \\ &-\int_0^1\int_0^1 \partial_{12}S(x_r,X_s)(X-W)(x-w)dsdr=\lambda(X-W)(x-w) \end{aligned}$$

for some strictly negative λ, by the positive twist condition and for $x_r=(1-r)w+rx$. □

The following is a watered down version of the Fundamental Lemma in Aubry & Le Daeron (1983). We follow Meiss (1992) :

Lemma 10.2 (Aubry's Fundamental Lemma) *Two distinct minimizers cross at most once.*

Proof. Suppose that $\boldsymbol{x}$ and $\boldsymbol{w}$ are two *distinct* minimizers who cross twice. We perform some surgery on finite segments of $\boldsymbol{x}$ and $\boldsymbol{w}$ to get two new sequences $\boldsymbol{x}'$ and $\boldsymbol{w}'$ with at least one of them of lesser action, contradicting minimality. There are three cases to consider: (i) both crossings are at non integers, (ii) one crossing is at an integer, (iii) both crossings are at integers.

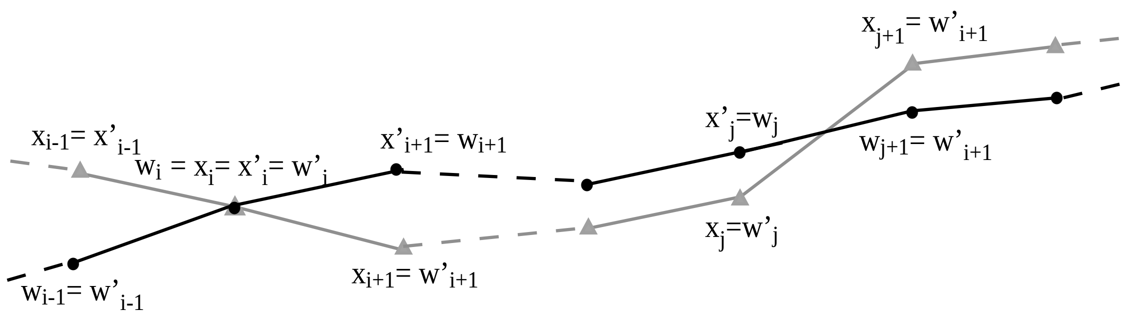

Fig. 10.2. A crossing of Case (ii)

Case (i): Let $t_1 \in (i-1, i)$ and $t_2 \in (j, j+1)$ be the crossing times. Define:

$$x'_k = \begin{cases} w_k & \text{if} \quad k \in [i,j] \\ x_k & \text{otherwise} \end{cases} \qquad w'_k = \begin{cases} x_k & \text{if} \quad k \in [i,j] \\ w_k & \text{otherwise} \end{cases}$$

Letting W denote the action over an interval $[N, M]$ containing $[j-1, k+1]$, we easily compute that:

$$\begin{aligned} W(\boldsymbol{x}') + W(\boldsymbol{w}') - W(\boldsymbol{x}) - W(\boldsymbol{w}) = & \\ & S(x_{i-1}, w_i) + S(w_{i-1}, x_i) - S(x_{i-1}, x_i) - S(w_{i-1}, w_i) \\ & + S(x_j, w_{j+1}) + S(w_j, x_{j+1}) - S(x_j, x_{j+1}) - S(w_j, w_{j+1}). \end{aligned}$$

The Crossing Lemma 10.1 shows that this difference of actions is negative, contradicting the minimality of $\boldsymbol{x}$ and $\boldsymbol{w}$.

Case (ii): In this case, only one crossing will contribute negatively to the difference of action of new and old sequences. We still get a contradiction.

Case (iii) Let $i-1$ and $j+1$ be the crossing times of $\boldsymbol{x}$ and $\boldsymbol{w}$, and construct $\boldsymbol{x}'$ and $\boldsymbol{w}'$ as before. In this case the difference in action between old and new segments is null. The sequences $\boldsymbol{x}', \boldsymbol{w}'$ must be minimizing, and hence correspond to orbits. But we have $x_{i-2} = w'_{i-2}, \quad x_{i-1} = w'_{i-1}$. Hence the points $\psi^{-1}(x_{i-2}, x_{i-1})$ and $\psi^{-1}(w'_{i-2}, w'_{i-1})$ of $\mathbb{R}^2$ are the same and thus generate the same orbit under F. This in turn implies that $\boldsymbol{x} = \boldsymbol{w}$, a contradiction to our assumption. □

Corollary 10.3 *W_{mn}-minimizing sequences are CO and their set is completely ordered for the partial order on sequences.*

Proof. Since the proof of Aubry's Lemma deals with finite segments of sequences only, it also applies to show that two W_{mn}-minimizers in $X_{m,n}$, may not cross twice within one period n. But two m, n-periodic sequences that cross once must necessarily cross twice within one period. Hence two W_{mn}-minimizers cannot cross at all. It is easy to check that W_{mn} is invariant under $\tau_{i,j}$ for all integers i, j. Thus, if $\boldsymbol{x}$ is a W_{mn} minimizer, $\tau_{i,j}\boldsymbol{x}$ is also a W_{mn}-minimizer. Since they do not cross, one must have either $\boldsymbol{x} \leq \tau_{i,j}\boldsymbol{x}$ or $\tau_{i,j}\boldsymbol{x} \leq \boldsymbol{x}$, for all $i, j \in \mathbb{Z}$, *i.e.* $\boldsymbol{x}$ is CO. □

We end this section by a proposition which we will need in Chapter 3.

Proposition 10.4 *Any W_{mn}-minimizer is a minimizer.*

Proof. We will show that if $\boldsymbol{x}$ is a W_{mn}-minimizer, it is also a W_{kmkn} minimizer for any k. This implies that $\boldsymbol{x}$ is a minimizer on segments of arbitrary length: if $\boldsymbol{x}$ is a W_{kmkn} minimizer, any segment of $\boldsymbol{x}$ of length less than kn is minimizing. Hence $\boldsymbol{x}$ is a minimizer. Now, take a W_{kmkn}-minimizer $\boldsymbol{w}$. If $\boldsymbol{w}$ is not m, n-periodic, then $\boldsymbol{w}$ and $\tau_{m,n}\boldsymbol{w}$ are distinct. By Corollary 10.3, they cannot cross. Suppose, say, that $\tau_{m,n}\boldsymbol{w} > \boldsymbol{w}$. Since $\tau_{m,n}$ trivially preserves the (strict) order on sequences, we must also have $\tau^k_{m,n}\boldsymbol{w} > \boldsymbol{w}$, a contradiction to the fact that $\boldsymbol{w}$ is km, kn- periodic. Hence $\boldsymbol{w}$ is in X_{mn} and its action over intervals of any length multiple of n cannot be less than that of $\boldsymbol{x}$. Hence $\boldsymbol{x}$ is also a W_{kmkn} minimizer. □

Exercise 10.5 Show that the set of minimizers (either sequences or orbits) is closed under pointwise limits.

Exercise 10.6 a) Show that the set of recurrent minimizers of rotation number ω is completely ordered. (*Hint.* Mimic the proof of Proposition 10.4: if an appropriate inequality is not satisfied, there must be a crossing. By recurrence, there is another one, a contradiction to Aubry's Lemma).
b) Show that a minimizer corresponding to a recurrent (not necessarily periodic) orbit of the twist map is CO.
(Remember that the orbit z_k of a dynamical system is called *recurrent* if z_0 is the limit of a subsequence z_{k_j}. Equivalently, z_0 is in its own ω–limit set).

11. CO Orbits of All Rotation Numbers

A. Existence of CO Periodic Orbits

We prove that the set of W_{mn}-minimizers is not empty. By Corollary 10.3 this will show the existence of CO orbits of all rational rotation numbers.

Proposition 11.1 *Let F be the lift of a twist map with a generating function which satisfies the coercion condition $\lim_{|X-x|\to\infty} S(x,X) \to +\infty$. Then, for all m, n, W_{mn} has a minimum on $X_{m,n}$.*

Proof. Note that, by periodicity of S, the ranges of W_{mn} on $X_{m,n}$ and on its subset $X_{m,n} \cap \{x_1 \in [0,1]\}$ are the same: we can translate any sequence of $X_{m,n}$ by an integer to bring it to that subset without changing its action. Now, if S satisfies the coercion condition, then for $\boldsymbol{x} \in X_{m,n} \cap \{x_1 \in [0,1]\}$, $\lim_{\|\boldsymbol{x}\|\to\infty} W_{mn}(\boldsymbol{x}) \to +\infty$: if $\|\boldsymbol{x}\| \to \infty$ and x_1 remains bounded, at least one $|x_k - x_{k-1}|$ must tend to $+\infty$. In particular, for any large enough $K \in \mathbb{R}$, $W_{mn}^{-1}(-\infty, K]$ is bounded and not empty. Since, by continuity, this set is also closed, it must be compact. Thus W_{mn} attains its minimum on that set. □

An interesting sufficient condition for S to satisfy the coercion condition is that the "twist" of the map be uniformly bounded below (see MacKay & al. (1989)):

Proposition 11.2 *Let the twist condition for the lift of a twist map F be uniform:*

$$\frac{\partial X(x,y)}{\partial y} > a > 0 \quad \forall (x,y) \in \mathbb{R}^2.$$

Then there is a constant α, and two strictly positive constants β and γ such that :

$$S(x,X) \geq \alpha - \beta\,|X-x| + \gamma\,|X-x|^2\,.$$

Proof. We can write:

$$S(x,X) = S(x,x) + \int_0^1 \partial_2 S(x,X_s)(X-x)ds,$$

where $X_s = (1-s)x + sX$. Applying the same process to $\partial_2 S$, we get:

$$S(x,X) = S(x,x) + \int_0^1 \partial_2 S(X_s, X_s)(X-x)ds \\ - \int_0^1 ds \int_0^1 \partial_{12} S(X_r, X_s)(X-x)^2 dr \quad .$$

We can conclude the proof of the lemma by taking

$$\alpha = \min_{x \in \mathbb{R}} S(x,x), \quad \beta = \max_{x \in \mathbb{R}} |\partial_2 S(x,x)|$$

(α, β exist by periodicity of S) and $\gamma = a/2$. □

B. Existence of CO Orbits of Irrational Rotation Numbers

The existence of CO orbits of irrational rotation numbers is a simple consequence of the existence of CO periodic orbits: pick a sequence $\boldsymbol{x}^{(k)}$ of W_{m_k,n_k}–minimizers, with $m_k/n_k \to \omega$ as $k \to \infty$. By using appropriate translations of the type $\tau_{m,0}$ on $\boldsymbol{x}^{(k)}$ (which neither change their rotation numbers, nor the fact that they are minimizers) we can assume that $x_0^{(k)} \in [0,1]$. The sequence m_k/n_k is bounded and hence, by Corollary 10.3 the sequences $\boldsymbol{x}^{(k)}$ are in $CO_{[a,b]} \cap \{\boldsymbol{x} \in \mathbb{R}^{\mathbb{Z}} \mid x_0 \in [0,1]\}$ for some $a, b \in \mathbb{R}$. Lemma 9.2 guarantees the existence of a converging subsequence in $CO_{[a,b]}$ and Lemma 9.1 shows that the limit of this subsequence has rotation number ω. Finally, note that the periods n_k go to infinity as k goes to infinity. In particular, any finite segment of a limit $\boldsymbol{x}$ of $\boldsymbol{x}^{(k)}$ is the limit of minimizing segments, hence minimizing itself (Exercise 10.5). □

12. Aubry-Mather Sets

We have proven Part (1) and (3) of the Aubry-Mather theorem: existence of cyclically ordered, minimizing orbits of all rotation numbers. We now prove Part (2): the cyclically ordered orbits that we found in the previous section lie on Aubry-Mather sets, which we describe in this section.

We say that a set M in $\mathbb{R}^2$ is F-*ordered* if, for z, z' in M,

$$\pi(z) < \pi(z') \Rightarrow \pi(F(z)) < \pi(F(z')),$$

where π is the x-projection. A set is F-*ordered invariant* if it is F-ordered and invariant under both F and F^{-1}. On such a set, the sequences $\boldsymbol{x}, \boldsymbol{x}'$ of x–coordinates of z and z'

must satisfy $\boldsymbol{x} \prec \boldsymbol{x}'$. An example of F-ordered invariant set is the set of points in a CO orbit and all their integer translates. In fact, this can be used to give an alternative definition of CO orbits: an orbit is CO if and only if its points form an F-ordered invariant set. Note that an invariant circle which is a graph is F-ordered invariant (we will see in Chapter 6 that all invariant circles are graphs). We now want to explore the properties of F-ordered invariant sets. Crucial to the properties of these sets is the following *ratchet phenomenon* (I owe this terminology to G.R. Hall), which is a somewhat quantitative expression of the twist condition. This phenomenon, or condition is best described by the following picture:

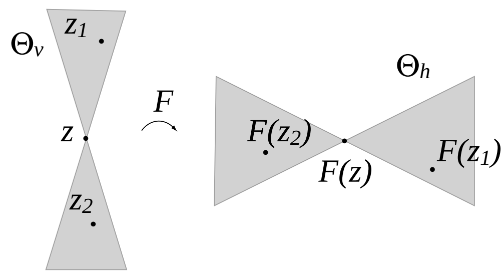

Fig. 12.0. The ratchet phenomenon for the lift of a positive twist map F: there are two cones (shaded in this picture) Θ_v and Θ_h in $\mathbb{R}^2$ centered around the y and x-axes respectively, such that, if z, z' are two points of $\mathbb{R}^2$ with $z' \in z + \Theta_v$, then $F(z') \in F(z) + \Theta_h$. More precisely, for a positive twist map $z' \in z + \Theta_v^+ \Rightarrow F(z') \in F(z) + \Theta_h^+$, where the half cones Θ_h^+, Θ_v^+ have the obvious meaning. The same holds for the half cones Θ_h^- and Θ_v^-. If g is *negative* twist (*eg.* F^{-1}), then the signs are reversed. The same cones can be used for F^{-1} as for F.

Lemma (Ratchet) 12.1 *Let F be the lift of a twist map satisfying $\frac{\partial X}{\partial y} > a > 0$ in some region. Then, in that region, F satisfies the ratchet phenomenon for some cones Θ_v, Θ_h whose angles only depend on a.*

Proof. See Exercise 12.9.

Proposition 12.2 *The closure of an F–ordered invariant set is F–ordered and invariant.*

Proof. The invariance is by continuity of F. Let M be an F-ordered invariant set. We let the reader prove that the uniform twist condition $\frac{\partial X}{\partial y} > a > 0$ is automatically satisfied on an

F-ordered invariant set (essentially, such a set is necessarily bounded in the y direction, see Exercise 12.9). Suppose that, in the closure $\overline{M}$ of M there are z, z' in $\overline{M}$, with $\pi(z) < \pi(z')$ but $\pi(F(z)) = \pi(F(z'))$ (the worst case scenario). By the ratchet phenomenon for F^{-1}, $F(z)$ must be above $F(z')$ and $\pi(F^2(z')) < \pi(F^2(z))$, $i.e.$ the x orbits of z and z' switched order. This is impossible since in M the (strict) order is preserved by F.

Proposition 12.3 *If M is an F-ordered invariant set, then it is a Lipschitz graph over its projection: there exists a constant K depending only on F such that, if (x, y) and (x', y') are two points of M, then:*

$$|y' - y| \leq K|x' - x|$$

with K only depending on the twist constant $a = \inf_M \frac{\partial X}{\partial y}$.

Note that a, and hence K could also be chosen the same for all F-ordered sets in a compact region.

Proof. The proof of Lemma 12.2 shows that if M is F-ordered, we cannot have z, z' in M and $\pi(z) = \pi(z')$ unless $z = z'$. Hence π is injective on M, and M is a graph. To show that M forms a *Lipschitz* graph over its projection, let z and z' be two points of M and $\boldsymbol{x}$ and $\boldsymbol{x}'$ the corresponding sequences of x-coordinates of their orbits. Assuming $\pi(z) < \pi(z')$, we must have $\boldsymbol{x} \prec \boldsymbol{x}'$. If $z' \in z + \Theta_v^+$, the ratchet phenomenon implies that $F^{-1}(z') \in F^{-1}(z) + \Theta_h^-$, $i.e.$ $x'_{-1} > x_{-1}$, a contradiction. Likewise z' cannot be in the cone $z + \Theta_v^-$, and hence it must be in the cone complementary to Θ_v at z. This cone condition is easily transcribed into a uniform Lipschitz condition $|y' - y| < K|x' - x|$. □

Remark 12.4 Applied to the special case of invariant circles, Proposition 12.3 shows that any invariant circle for a twist map which is a graph is Lipschitz. This is a theorem originally due to Birkhoff, who also proved (see Chapter 6) that all non-homotopically trivial invariant circles for twist maps must be graphs.

Lemma 12.5 *All points in an F-ordered set have the same rotation number.*

Proof. This is a consequence of the simple fact (Lemma 13.3 in the appendix) that if $\boldsymbol{x} < \boldsymbol{x}'$ are two CO sequences, they must have the same rotation number. □

Definition 12.6 An *Aubry-Mather set* M for the lift F of a twist map f of the cylinder is a closed, F-ordered invariant set which is also invariant under the integer translation T.

Note that some authors call Aubry-Mather sets the projections of the above sets to the annulus. Exercise 12.9 shows that these projections are necessarily compact. Taking the closure of all the integer translates of the points in the CO orbits found in the previous section, we immediately get:

Theorem 12.7 *Let F be the lift of a twist map of the cylinder. Then F has Aubry-Mather sets of all rotation numbers in $\mathbb{R}$. Any CO orbit is in an Aubry-Mather set.*

Note that this theorem gives part (b) of the Aubry-Mather theorem.

Theorem 12.8 (Properties of Aubry-Mather sets) *Let M be an Aubry-Mather set for a lift F of a twist map of the cylinder.*

(a) M forms a graph over its projection $\pi(M)$, which is Lipschitz with Lipschitz constant only depending on the twist constant $a = \inf_M \frac{\partial X}{\partial y}$.

(b) All the orbits in M are cyclically ordered and they all have the same rotation number, which is called the rotation number of M.

(c) The projection $\pi(M)$ is a closed invariant set for the lift of a circle homeomorphism, and hence F restricted to M is conjugated to the lift of a circle homeomorphism via π.

Proof of Theorem 12.8. We have shown in Lemmas 12.5 and 12.6 that (a) and (b) are in fact properties of F-ordered invariant sets. As for Property (c), since π is one to one on M, F induces a continuous (Lipschitz, in fact) increasing map G on $\pi(M)$, defined by $G(\pi(z)) = \pi(F(z)$. Since M and thus $\pi(M)$ are invariant under integer translation, we have $G(x+1) = G(x) + 1$. The set $\pi(M)$ is closed and invariant under integer translation since M is. If $\pi(M) = \mathbb{R}$, then G is the lift of a circle homeomorphism. If $\pi(M) \neq \mathbb{R}$, then its complement is made of open intervals. Extend G by linear interpolation on each interval in the complement of $\pi(M)$. Since G is increasing on $\pi(M)$, its extension to $\mathbb{R}$ (call it G) is increasing as well, continuous and $G(x+1) = G(x) + 1$, hence the lift of a circle homeomorphism. By construction $G(\pi(z)) = \pi(F(z))$, and $\pi\big|_M$ is a continuous, 1-1 map on the compact set M, hence a homeomorphism $M \to \pi(M)$. Thus π is a conjugacy between F on M and G on $\pi(M)$, which is a closed and invariant set under G and G^{-1}. □

Recapitulation on the Dynamics of Aubry-Mather Sets. If G is the lift of a circle homeomorphism constructed in the proof of Theorem 12.7, the possible dynamics for invariant sets of circle maps described in the appendix become, under the conjugacy, possible dynamics on Aubry-Mather sets M for F. Hence an Aubry-Mather set M is either:

(i) an ordered collection of periodic orbits with (possibly) heteroclinic orbits joining them, or

(ii) the lift of an f-invariant circle, or

(iii) an F-invariant Cantor set with (possibly) homoclinic orbits in its gaps.

The rotation number of M is necessarily rational in Case (i), and necessarily irrational in Case (iii). In Case (ii), M may either have a rational or irrational rotation number, as the example of the shear map shows. However, maps with rational invariant circles are non generic. Indeed, as a circle map, the restriction of the twist map to the invariant circle must have a periodic orbit. For generic twist maps, periodic orbits must be hyperbolic and the circle must be made of stable and unstable manifolds of such orbits, that coincide. But generically, such manifolds intersect transversally. See Herman (1983) and Robinson (1970) for more details. As for homoclinic and heteroclinic orbits as in (i) and (iii), they have been shown to exist each time there are no invariant circles of the corresponding rotation numbers, see Hasselblat & Katok (1995) , Mather (1986) .

The feature that is striking in the Aubry-Mather theorem is the possible occurrence of Aubry-Mather sets as in (iii). The F-invariant Cantor sets have been called *Cantori* by Percival (1979)who constructed them for the discontinuous sawtooth map (a standard map with sawtooth shaped potential). This type of dynamics does occur in twist map, since it can be shown that many maps have no invariant circles, and hence the irrational Aubry-Mather sets must be of type (iii), $i.e.$ contain Cantori.

Although one can construct many Aubry-Mather sets that are not made of minimizers (Mather (1985)), the name "Aubry Mather set" is often reserved to the action minimizing Cantori M_ω as defined below:

Proposition 12.9 *For each irrational rotation number ω there is a unique Cantorus M_ω made of recurrent minimizing orbits of rotation number ω. The closure of any CO minimizing orbit of rotation number ω is contained in M_ω.*

Proof. A CO minimizing orbit forms an F-ordered set, contained in an Aubry-Mather set, and hence conjugated to an orbit of a circle homeomorphism. The closure of the irrational CO minimizing orbit is therefore in a Cantorus, conjugated to the ω–limit set of the circle homeomorphism. As limit of minimizers, this Cantorus is made up of minimizers. We now prove that this Cantorus is unique: suppose there are two of them. Exercise 10.6 implies that the (disjoint) union of these two Cantori forms an F–ordered set, hence conjugated to a closed invariant set of a circle homeomorphism. Each Cantorus is the ω-limit set of its points. This is a contradiction to the uniqueness of ω limit sets of circle homeomorphisms proven in Theorem 13.4. □

Exercise 12.9 a) Prove the Ratchet Lemma 12.1.
b) Prove that if F is an F-ordered invariant set, then the projection $proj(M)$ of M to the cylinder is compact, f-invariant. Deduce from this that M satisfies the uniform twist condition $\partial X/\partial y > a > 0$. [*Hint.* Use Lemma 9.2].

Exercise 12.10 Show that a twist map f restricted to a Cantorus (irrational Aubry-Mather set) is semiconjugate to a rotation of the same rotation number.

13. Appendix: Cyclically Ordered Sequences and Circle Maps

In this section, we prove Lemma 9.1, and Lemma 9.2. We then recover important facts about circle homeomorphisms and their invariant sets using the language of CO sequences.

A. Proofs of Lemmas 9.1 and 9.2

We recall the statements of each lemma before proving it. Part of the proof below is classical, due to Poincaré in his study of circle homeomorphisms.

Lemma 9.1 *Let* $\{x_k\}_{k\in\mathbb{Z}}$ *be a CO sequence then* $\rho(\boldsymbol{x}) = \lim_{k\to\infty} x_k/k$ *exists and:*

$$|x_k - x_0 - k\rho(\boldsymbol{x})| \leq 1. \tag{13.1}$$

Moreover $\boldsymbol{x} \to \rho(\boldsymbol{x})$ *is a continuous function on CO sequences, when the set of sequences has been given the topology of pointwise convergence.*

Proof . Let $\boldsymbol{x}$ be a CO sequence. We first prove that the sequence $\{\frac{x_n - x_0}{n}\}_{n\in\mathbb{Z}}$ is Cauchy as $n \to \pm\infty$. We do the case $n \to +\infty$ first.

Given $n \in \mathbb{N}$, let α_n be the integer such that:

$$x_0 + \alpha_n \leq x_n < x_0 + \alpha_n + 1. \tag{13.2}$$

We prove by induction that

$$x_0 + k\alpha_n \leq x_{kn} < x_0 + k\alpha_n + k, \quad \forall k \in \mathbb{N}. \tag{13.3}$$

Indeed, step 1 in the induction is just (13.2) , and if we assume step k, $i.e.$ (13.3) then, since $\boldsymbol{x}$ is CO, we get

$$x_n + k\alpha_n \leq x_{(k+1)n} < x_n + k\alpha_n + k.$$

Using (13.2) this gives $x_0 + (k+1)\alpha_n \leq x_{(k+1)n} < x_0 + (k+1)\alpha_n + (k+1)$, which is the step $k+1$ and finishes the induction.

Dividing (13.3) by k we get

$$\alpha_n \leq \frac{x_{kn} - x_0}{k} < \alpha_n + 1. \tag{13.4}$$

Since this is true for all $k > 0$, we must have, for all $n \neq 0$, the two equivalent inequalities

$$\left|\frac{x_{kn} - x_0}{k} - \frac{x_n - x_0}{1}\right| \leq 1 \Leftrightarrow \left|\frac{x_{kn} - x_0}{kn} - \frac{x_n - x_0}{n}\right| \leq \frac{1}{|n|}. \tag{13.5}$$

Writing $z_n = \frac{x_n - x_0}{n}$, and assuming $m > 0, n > 0$, the triangular inequality gives:

$$|z_n - z_m| \leq |z_n - z_{mn}| + |z_{mn} - z_m| \leq \frac{1}{n} + \frac{1}{m}, \tag{13.6}$$

and hence $\{z_n\}_{n\in\mathbb{N}}$, is a Cauchy sequence whose limit we call $\rho(\boldsymbol{x})$.

Let $m \to \infty$ in (13.6), and multiply by n:

$$|x_n - x_0 - n\rho(\boldsymbol{x})| \leq 1. \tag{13.7}$$

To see how the case $n \to -\infty$ follows, note that in all the above we could have replaced x_0 by an arbitrary $x_m, m \in \mathbb{Z}$ and obtained:

$$|x_n - x_m - (n-m)\rho(\boldsymbol{x})| \leq 1 \qquad \forall m, n \in \mathbb{Z}. \tag{13.8}$$

We let the reader check that this last inequality implies that $\lim_{n\to-\infty} z_n = \rho(\boldsymbol{x})$.

The continuity of ρ is also a consequence of (13.7). Indeed, suppose the CO sequences $\boldsymbol{x}^{(j)}$ tend to $\boldsymbol{x}$ pointwise as $j \to \infty$. Constructing sequences $\boldsymbol{z}^{(j)}$ as above, and denoting $\rho(\boldsymbol{x}^{(j)}) = \omega_j$, (13.7) yields

$$|z_k^{(j)} - \omega_j| \leq \frac{1}{k}, \quad |z_k - \rho(\boldsymbol{x})| \leq \frac{1}{k}. \tag{13.9}$$

Since $\boldsymbol{z}^{(j)} \to \boldsymbol{z}$, for all k and $\epsilon > 0$,

$$|\omega_j - \omega_i| \leq |\omega_j - z_k^{(j)}| + |z_k^{(j)} - z_k^{(i)}| + |z_k^{(i)} - \omega_i| \leq \frac{2}{k} + \epsilon$$

whenever i, j are big enough. Hence $\{\omega_k\}_{k\in\mathbb{Z}}$ is a Cauchy sequence, whose limit we denote by ω. Letting $j \to \infty$ in (13.9) yields $\omega = \rho(x)$. □

Lemma 13.1 *The sets* $CO_{[a,b]}/\tau_{1,0}$ *and* $CO_{[a,b]} \cap \{\boldsymbol{x} \in \mathbb{R}^{\mathbb{Z}} \mid x_0 \in [0,1]\}$ *are compact for the topology of pointwise convergence.*

Proof. We have already remarked that, trivially, CO is closed for pointwise convergence, *i.e.* the product topology on sequences. Lemma 9.1 implies that $\mathrm{CO}_{[a,b]} \cap \{\boldsymbol{x} \mid x_0 \in [0,1]\}$ is a closed subset of the set:

$$\{\boldsymbol{x} \in \mathbb{R}^{\mathbb{Z}} \mid x_k = x_0 + k\omega + y_k, (x_0, \omega, \boldsymbol{y}) \in [0,1] \times [a,b] \times [-1,1]^{\mathbb{Z}}, \text{with } y_0 = 0\}$$

which is compact for the product topology, by Tychonov's theorem. We let the reader derive a similar proof for $\mathrm{CO}_{[a,b]}/\tau_{1,0}$. □

B. Dynamics of Circle Homeomorphisms

Rotation Numbers and Circle Homeomorphisms. The orbits of an orientation preserving circle homeomorphism are by definition Cyclically Ordered. From Lemma 9.1, we can deduce the following theorem, due to Poincaré (1885):

Theorem 13.2 *All the orbits of the lift F of an orientation preserving circle homeomorphism f have the same rotation number, denoted by $\rho(F)$. The rotation number ρ is a continuous function of F, where the set of lifts of homeomorphisms of the circle is given the C^0 topology.*

Proof. We start by a simple but useful lemma.

Lemma 13.3 *If two* CO *sequences $\boldsymbol{x}, \boldsymbol{x}'$ satisfy $\boldsymbol{x} < \boldsymbol{x}'$ then $\rho(\boldsymbol{x}) = \rho(\boldsymbol{x}')$.*

Proof. The rotation numbers are the respective asymptotic slopes of the Aubry diagram of $\boldsymbol{x}$ and $\boldsymbol{x}'$. Thus, if $\rho(\boldsymbol{x}) \neq \rho(\boldsymbol{x}')$, the Aubry diagrams of $\boldsymbol{x}$ and $\boldsymbol{x}'$ must cross. In this case, there must be a k_0 and a k_1 such that $x_{k_0} > x'_{k_0}$ and $x_{k_1} < x'_{k_1}$. This contradicts $\boldsymbol{x} < \boldsymbol{x}'$. □

Continuing with the proof of Theorem 13.2, since F is increasing, two CO sequences $\boldsymbol{x}$ and $\boldsymbol{w}$ corresponding to distinct orbits of F must satisfy $\boldsymbol{x} \prec \boldsymbol{w}$ or $\boldsymbol{w} \prec \boldsymbol{x}$. From the previous lemma $\boldsymbol{x}$ and $\boldsymbol{w}$ have same rotation number. Finally, if $f_n \to f$ in the C^0 topology, then the f_n orbit of a point x (a CO sequence) tends pointwise to the f orbit of x. By Lemma 9.1, $\lim \rho(f_n) = \lim \rho(\{f_n^k(x)\}_{k\in\mathbb{Z}}) = \rho(\{f^k(x)\}_{k\in\mathbb{Z}}) = \rho(f)$. □

Dynamical classification of circle homeomorphisms. We now review the classification of circle homeomorphisms by Poincaré (1885). Recall some general terminology from dynamical systems. The *Omega limit set* $\omega(x)$ of a point x under a dynamical system f is the set of limit points of the forward orbit, $i.e.$ the set of limit points of all subsequences $\{x_{k_j}\}$ where $x_k = f^k(x)$ and $k_j \to +\infty$ as $j \to +\infty$. Likewise, the *Alpha limit set* $\alpha(x)$ is the set of limit points of the backward orbit. A *minimal invariant set* for a dynamical system is a closed, (forward and backward) invariant set which contains no closed invariant proper subset. A *heteroclinic orbit* between two invariant sets A and B is the orbit of a point x such that $\alpha(x) \subset A$ and $\omega(x) \subset B$. The term *homoclinic* is used when $A = B$.

Theorem 13.4 *Let f be a circle homeomorphism and F a lift of f. If $\rho(F)$ is rational, then, for any $x \in \mathbb{S}^1$, $\omega(x)$ and $\alpha(x)$ are periodic orbits. The orbit of x is either periodic (in which case $x \in \omega(x) = \alpha(x)$) or it is heteroclinic between $\alpha(x)$ and $\omega(x)$.*

If $\rho(F)$ is irrational, then, for any $x, x' \in \mathbb{S}^1$, $\alpha(x) = \alpha(x') = \omega(x) = \omega(x')$. Call this set $\Omega(f)$. Then $\Omega(f)$ is either the full circle, or a minimal invariant set which is a Cantor set. In the first case any orbit is dense in the circle, and f is conjugated to a rotation by $\rho(F)$. In the second case, a point x of $\mathbb{S}^1$ is either in $\Omega(f)$ and recurrent, or it is homoclinic to $\Omega(f)$, a "gap orbit", and f is semi-conjugate to a rotation by $\rho(F)$.

We remind the reader that a *Cantor set* K is a closed, perfect, and totally disconnected topological set. *Perfect* means that each point in K is the limit of some (not eventually constant) sequence in K, and *totally disconnected* means that, given any two points a and b in K, one can find disjoint closed sets A and B with $a \in A, b \in B$ and $A \cup B = K$. In the real line or the circle, a closed set is totally disconnected if and only if it is nowhere dense. A set X is *nowhere dense* if $Interior(Closure(X)) = \emptyset$.

Proof of Theorem 13.4.

Rational rotation number. Suppose $\rho(F) = m/n$. Then $F^n(\cdot) - m$ must have a fixed point, otherwise for all $x \in \mathbb{R}$, $F^n(x) - x \neq m$ and we can assume $F^n(x) - x > m$. By compactness of $\mathbb{S}^1$, $\rho(F) > m/n$, a contradiction. Hence F has an m, n-periodic orbit. By continuity, on any interval I where $F^n - Id - m$ is non zero, it must stay of a constant sign. This sign describes the direction of progress of points inside the orbit of I towards its endpoints: they must be heteroclinic to the endpoint orbits. Conversely, if F has an m, n-periodic orbit, its rotation number and thus that of F must be m/n.

Irrational rotation number. Suppose $\rho(F)$ is irrational. Let $x \in \mathbb{S}^1$ and denote by $\boldsymbol{x} = \{x_k\}_{k \in \mathbb{Z}}$ its orbit under f (with $x = x_0$). Suppose $\omega(x) = \mathbb{S}^1$. We show that $\omega(x') = \mathbb{S}^1$ for any other $x' \in \mathbb{S}^1$. Suppose not, and there is an interval (a, b) which contains no $x'_k = f^k(x')$. But (a, b) must contain some $[x_n, x_m]$ by density of $\boldsymbol{x}$. Again by density, the intervals $f^{-i(m-n)}[x_n, x_m]$ must cover $\mathbb{S}^1$ and hence $f^{i(m-n)}x' \in (a, b)$ for some i, a contradiction. We guide the reader through the proof that f is conjugated to a rotation by $\rho(f)$ in Exercise 13.6.

Suppose $\omega(x) \neq \mathbb{S}^1$. Then, since $\omega(x)$ is closed, its complement is the union of open intervals. Take another point x'. We want to show that $\omega(x') = \omega(x)$. We will prove that $\omega(x') \subset \omega(x)$: by symmetry $\omega(x) \subset \omega(x')$. This is obvious if $x' \in \omega(x)$. Suppose not. Then x' is in an open interval I in the complement of $\omega(x)$ whose endpoints are in $\omega(x)$. The orbit of I is made of open intervals in the complement of $\omega(x)$ whose endpoints are orbits in $\omega(x)$. Since there is no periodic orbit, these intervals are disjoint: by the intermediate value theorem $f^k(I) \subset I$ would imply the existence of a fixed point for f^k, hence a periodic orbit. The length of these intervals must tend toward 0 under iteration. Thus the orbit of x' approaches the endpoint orbit of I arbitrarily $i.e.$ the orbit of x' is asymptotic to $\omega(x)$. Hence $\omega(x') \subset \omega(x)$. In particular $\omega(x) = \Omega(f)$ is a minimal invariant set: any closed invariant subset of $\Omega(f)$ must contain the ω-limit set of any of its point, hence $\Omega(f)$ itself.

We now show that $\Omega(f)$ is a Cantor set. That it is closed is a property of ω-limit sets. It is perfect since $x \in \Omega(f)$ means that $x \in \omega(x)$ and hence $f^{n_k}(x) \to x$ for some $n_k \nearrow \infty$ and the $f^{n_k}(x)$'s are in $\omega(x)$, and are all distinct. To prove that $\Omega(f)$ is nowhere dense, first note that the topological boundary $\partial\Omega(f) = \Omega \backslash Interior(\Omega(f))$ must satisfy $\partial\Omega(f) = \Omega(f)$ or $\partial\Omega(f) = \emptyset$: $\partial\Omega(f)$ is closed, invariant under f and included in $\Omega(f)$ which is a minimal set. But $\partial\Omega(f) = \emptyset$ means $\Omega(f) = Interior(\Omega(f))$ is open, and because it is also closed, it must be all of $\mathbb{S}^1$, which we have ruled out. The alternative is $\partial\Omega(f) = \Omega(f)$, which means $Interior(\Omega(f)) = \emptyset$, which is what we wanted to prove. Exercise 13.6 walks the reader through the proof that f is semi-conjugate to a rotation in this case. □

Remark 13.5 A circle homeomorphism with an invariant Cantor set cannot be too smooth: Denjoy (see Hasselblat & Katok (1995), Robinson (1994)) proved that if f is a C^1 diffeomorphism of $\mathbb{S}^1$ with irrational rotation number and derivative of bounded variation, then f has a dense orbit ($i.e.$ $\Omega(f) = \mathbb{S}^1$) and is therefore conjugated to a rotation of angle $\rho(F)$. On the other hand, Denjoy did construct a C^1 diffeomorphism with $\Omega(f)$ a Cantor set. The idea is simple: take a rotation by irrational angle α. Cut the circle at some point x and at all its iterate $f^k(x)$. Glue in at these cuts intervals I_k of length going to 0 as $k \to \infty$, in such a way that the new space you obtain is again a circle. Extend the map f by linear interpolation on the I_k. You get a circle homeomorphism with rotation number α. With some care, one can make this homeomorphism differentiable, but only up to a point (C^1 with Hölder derivative). The complement of the I_k's in the new circle is a Cantor set, which is minimal for the extended map.

Exercise 13.6 In this exercise, we prove that all orientation preserving circle homeomorphism with irrational rotation number ω has the rotation of angle ω as a factor. This is sometimes called Poincaré's Classification Theorem (see Hasselblat & Katok (1995)).
a) Prove that $\boldsymbol{x}$ is a CO sequence with irrational $\rho(\boldsymbol{x})$ *iff*

$$\forall n, m, p \in \mathbb{Z}, \quad x_n < x_m + p \Longleftrightarrow n\rho(\boldsymbol{x}) < m\rho(\boldsymbol{x}) + p$$

(*Hint.* Use Formula (13.8) for multiples of m and n). What is the proper corresponding statement for CO sequences of rational rotation number?
b) Suppose the circle homeomorphism f has a dense orbit, which lifts to an orbit $\boldsymbol{x}$ of some F. Build a monotone map $h : \mathbb{R} \to \mathbb{R}$ by first defining it on $\boldsymbol{x}$ by:

$$x_k + m \mapsto k\rho(\boldsymbol{x}) + m, \quad \forall m, k \in \mathbb{Z}.$$

Use a) to show that h is order preserving and show that its extension by continuity is well defined, has continuous inverse and preserves orbits of F, and it commutes with the translation T (*Hint.* density of the orbit in $\mathbb{S}^1$ means density of the set $\{x_k + m\}_{k.m\in\mathbb{Z}}$ in $\mathbb{R}$). Hence in this case f is conjugate to a rotation.
c) Suppose now that $\Omega(f) \neq \mathbb{S}^1$. Following the steps in b), take a dense orbit $\boldsymbol{x}$ in $\Omega(f)$ and build a map $h : \Omega(F) \to \mathbb{R}$ as before ($\Omega(F)$ denotes the lift of $\Omega(f)$ here). Check that this map is onto, non decreasing and extend it to a map $\mathbb{R} \to \mathbb{R}$ by mapping each the gap of the Cantor set to a single point.
d) Conclude that, in both cases, h provides a (semi)-conjugacy between f and a rotation by ω.