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GENERATING PHASES AND SYMPLECTIC TOPOLOGY

In Appendix 1, Section 58, we remark that the differential of a function $W : M \to T^*M$ gives rise to the Lagrangian submanifold $dW(M)$ of T^*M. As a generalization of this fact, one can construct Lagrangian submanifolds of T^*M as symplectic reductions of graphs of differentials of generating phases, which are functions on vector bundles over M.

Generating phases are the common geometric framework to the different discrete variational methods in Hamiltonian systems, including the method developed in this book. Applications of generating phases range from the search for periodic orbits to the Maslov index, symplectic capacities and singularities theory. Generating phases are a viable alternative to the use of heavy functional analytic variational methods in symplectic topology.

This chapter intends to be a basic introduction to generating phases. We first present Chaperon's method, which he used to give an alternate proof of the theorem of Conley & Zehnder (1983). This theorem, which solved a conjecture by Arnold on the minimum number of periodic points of Hamiltonian maps of $\mathbb{T}^{2n}$, is considered by many as the starting point of symplectic topology, in the sense that it implies that the C^0 closure of the set of symplectic diffeomorphisms is distinct from the set of volume preserving diffeomorphisms. We then survey the abstract structure of generating phase, highlighting the common geometric frame for the symplectic twist maps method and that of Chaperon (as well as others).

52. Chaperon's Method And The Theorem Of Conley-Zehnder

Chaperon (1984) introduced a method "du type géodesiques brisées" for finding periodic orbits of Hamiltonians which did not make use of a decomposition by symplectic twist maps. This method has been the basis of later work by *eg.* Laudenbach & Sikorav (1985), Sikorav (1986), and Viterbo (1992).

A. A New Action Function

Until now, we have studied exact symplectic maps that come equipped with a generating function due to the twist condition. The concept of generating function is more general than this, however: we now show how an exact symplectic map of $\mathbb{R}^{2n}$ *which is uniformly* C^1 *close to* Id may have another kind of generating function. The small time t map of a large class of Hamiltonians satisfy this condition. Hence, the time one map of these Hamiltonians can be decomposed into maps that possess this kind of generating function, leading to a different variational setting for periodic orbits than the one we have used so far. Let

$$F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$
$$(\boldsymbol{q}, \boldsymbol{p}) \to (\boldsymbol{Q}, \boldsymbol{P})$$

be an exact symplectic diffeomorphism:

$$\boldsymbol{P} d\boldsymbol{Q} - \boldsymbol{p} d\boldsymbol{q} = F^* \boldsymbol{p} d\boldsymbol{q} - \boldsymbol{p} d\boldsymbol{q} = dS, \tag{52.1}$$

for some $S : \mathbb{R}^{2n} \to \mathbb{R}$ (remember that all symplectic diffeomorphism of $\mathbb{R}^{2n}$ are in fact exact symplectic. We stress *exact* symplectic here in view of our later generalization to T^*M). The following simple lemma is crucial here.

Lemma 52.1 *Let* $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *be an exact symplectic diffeomorphism. Then, if* $\|F - Id\|_{C^1}$ *is small enough, the map*

$$\phi : (\boldsymbol{q}, \boldsymbol{p}) \to (\boldsymbol{Q}, \boldsymbol{p})$$

is a diffeomorphism of $\mathbb{R}^{2n}$ *.*

Proof. $\boldsymbol{Q}(\boldsymbol{q}, \boldsymbol{p})$ is (uniformly) C^1 close to $\boldsymbol{q}$ and thus ϕ is (uniformly) C^1 close to Id, hence a diffeomorphism. □

Treating ϕ as a change of coordinates, we can view S as a function of the variables $\boldsymbol{Q}, \boldsymbol{p}$. We now show how, in a way that is slightly different from the twist map case, F can be recovered from S. We define

$$\tilde{S}(\boldsymbol{Q}, \boldsymbol{p}) = \boldsymbol{p}\boldsymbol{q} + S(\boldsymbol{Q}, \boldsymbol{p}), \quad \text{where} \quad \boldsymbol{q} = \boldsymbol{q}(\boldsymbol{Q}, \boldsymbol{p});$$

then

$$d\tilde{S} = \boldsymbol{p}d\boldsymbol{q} + \boldsymbol{q}d\boldsymbol{p} + \boldsymbol{P}d\boldsymbol{Q} - \boldsymbol{p}d\boldsymbol{q} = \boldsymbol{P}d\boldsymbol{Q} + \boldsymbol{q}d\boldsymbol{p} \tag{52.2}$$

and thus $\tilde{S}$ generates F, in the sense that:

$$\begin{aligned} \boldsymbol{P} &= \frac{\partial \tilde{S}}{\partial \boldsymbol{Q}}(\boldsymbol{Q}, \boldsymbol{p}) \\ \boldsymbol{q} &= \frac{\partial \tilde{S}}{\partial \boldsymbol{p}}(\boldsymbol{Q}, \boldsymbol{p}). \end{aligned} \tag{52.3}$$

Remark 52.2 Note that Id is not a symplectic twist map and thus it cannot be given a generating function in the twist map sense. One of the advantages of the present approach is that Id does have a generating function, which is

$$\tilde{S}(\boldsymbol{Q}, \boldsymbol{p}) = \boldsymbol{p}\boldsymbol{Q}.$$

As an illustration, fixed points of F are given by the equations:

$$\begin{aligned} \boldsymbol{p} &= \frac{\partial \tilde{S}}{\partial \boldsymbol{Q}} = \boldsymbol{P}, \\ \boldsymbol{Q} &= \frac{\partial \tilde{S}}{\partial \boldsymbol{p}} = \boldsymbol{q}, \end{aligned}$$

which are equivalent to the following equation:

$$d(\tilde{S} - \boldsymbol{p}\boldsymbol{Q}) = (\boldsymbol{P} - \boldsymbol{p})d\boldsymbol{Q} + (\boldsymbol{q} - \boldsymbol{Q})d\boldsymbol{p} = 0.$$

Hence we have reduced the problem of finding fixed points of an exact symplectic diffeomorphism C^1 close to Id on $\mathbb{R}^{2n}$ to the one of finding critical points for a real valued function. We now apply this method to give Hamiltonian maps of $\mathbb{T}^{2n}$ a finite dimensional variational context. It can also be used for time one maps of Hamiltonians with compact support in $\mathbb{R}^{2n}$, or Hamiltonian maps that are C^0 close to Id in a compact symplectic manifold.

Let $H : \mathbb{R}^{2n} \times \mathbb{R}$ be a C^2 function with variables $(\boldsymbol{q}, \boldsymbol{p}, t)$. Assume H is $\mathbb{Z}^{2n}$ periodic in the variables $(\boldsymbol{q}, \boldsymbol{p})$ (i.e., H is a function on $\mathbb{T}^{2n} \times \mathbb{R}$). As in Appendix 1, we denote by $h_{t_0}^t(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q}(t), \boldsymbol{p}(t))$ the solution of Hamilton's equations with initial conditions $\boldsymbol{q}(t_0) = \boldsymbol{q}, \quad \boldsymbol{p}(t_0) = \boldsymbol{p}$. By assumption, $h_{t_0}^t$ can be seen as a Hamiltonian map on $\mathbb{T}^{2n}$. We know that $h_{t_0}^t$ is exact symplectic (see Theorem 59.7). Furthermore, by compactness of $\mathbb{T}^{2n}$, *when* $|t - t_0|$ *is small,* $h_{t_0}^t$ *is* C^1 *close to* Id (the Hamiltonian vector field of a C^2 function is C^1, hence so is its flow). For $|t - t_0|$ small enough, we can apply Lemma 52.1 to get a generating function for $h_{t_0}^t$. To make this argument global, we decompose h^1 in smaller time maps:

$$h^1 = h^1_{\frac{N-1}{N}} \circ h^{\frac{N-1}{N}}_{\frac{N-2}{N}} \circ \ldots \circ h^{\frac{2}{N}}_{\frac{1}{N}} \circ h_0^{\frac{1}{N}} \tag{52.4}$$

and thus, for a large enough N, h^1 can be decomposed into N maps that satisfy Lemma 52.1. [The farther h^1 is from Id, the bigger N must be]. We can then apply the following proposition to h^1:

Proposition 52.3 *Let* $F = F_N \circ \ldots \circ F_1$ *where each* F_k *is exact symplectic in* $T^*\mathbb{R}^n$*,* C^1 *close to* Id*, and has generating function* $\tilde{S}_k(\boldsymbol{Q}, \boldsymbol{p})$*. The fixed points of* F *are in one to one correspondence with the critical points of :*

$$\tilde{W}\left(\boldsymbol{Q}_1, \boldsymbol{p}_1, \ldots, \boldsymbol{Q}_N, \boldsymbol{p}_N\right) = \sum_{k=1}^{N} \tilde{S}_k(\boldsymbol{Q}_k, \boldsymbol{p}_k) - \boldsymbol{p}_k \boldsymbol{Q}_{k-1}$$

where we set $\boldsymbol{Q}_0 = \boldsymbol{Q}_N$*.*

Proof. We will use the usual notation

$$(\boldsymbol{P}_k, \boldsymbol{Q}_k) = F_k(\boldsymbol{q}_k, \boldsymbol{p}_k)$$

where we know from (52.3) that $\boldsymbol{P}_k$ and $\boldsymbol{q}_k$ are functions of $\boldsymbol{Q}_k, \boldsymbol{p}_k$. Then, using Equation (52.2),

$$
\begin{aligned}
d\tilde{W}(\overline{\boldsymbol{Q}}, \overline{\boldsymbol{p}}) &= \sum_{k=1}^{N} \boldsymbol{P}_k d\boldsymbol{Q}_k + \boldsymbol{q}_k d\boldsymbol{p}_k - \boldsymbol{p}_k d\boldsymbol{Q}_{k-1} - \boldsymbol{Q}_{k-1} d\boldsymbol{p}_k \\
&= \sum_{k=1}^{N-1} (\boldsymbol{P}_k - \boldsymbol{p}_{k+1}) d\boldsymbol{Q}_k + \sum_{k=2}^{N} (\boldsymbol{q}_k - \boldsymbol{Q}_{k-1}) d\boldsymbol{p}_k \\
&\quad + (\boldsymbol{P}_N - \boldsymbol{p}_1) d\boldsymbol{Q}_N + (\boldsymbol{q}_1 - \boldsymbol{Q}_N) d\boldsymbol{p}_1.
\end{aligned} \tag{52.5}
$$

This formula proves that $(\overline{\boldsymbol{Q}}, \overline{\boldsymbol{p}})$ is critical exactly when:

$$
\begin{aligned}
&F_k(\boldsymbol{q}_k, \boldsymbol{p}_k) = (\boldsymbol{q}_{k+1}, \boldsymbol{p}_{k+1}), \forall k \in \{1, \ldots, N-1\}, \\
&F_N(\boldsymbol{q}_N, \boldsymbol{p}_N) = (\boldsymbol{q}_1, \boldsymbol{p}_1),
\end{aligned}
$$

that is, exactly when $(\boldsymbol{q}_1, \boldsymbol{p}_1)$ is a fixed point for F. □

B. Interpretation Of $\tilde{W}$ As Action Of Broken Geodesic

When F is the time 1 map of some Hamiltonian and we decompose F as in (52.4), $\tilde{W}$ has the interpretation of the action of a "broken" solution of the Hamiltonian equation. This is similar to the situation in Chapter 7. This time however, the jumps are both vertical *and* horizontal:

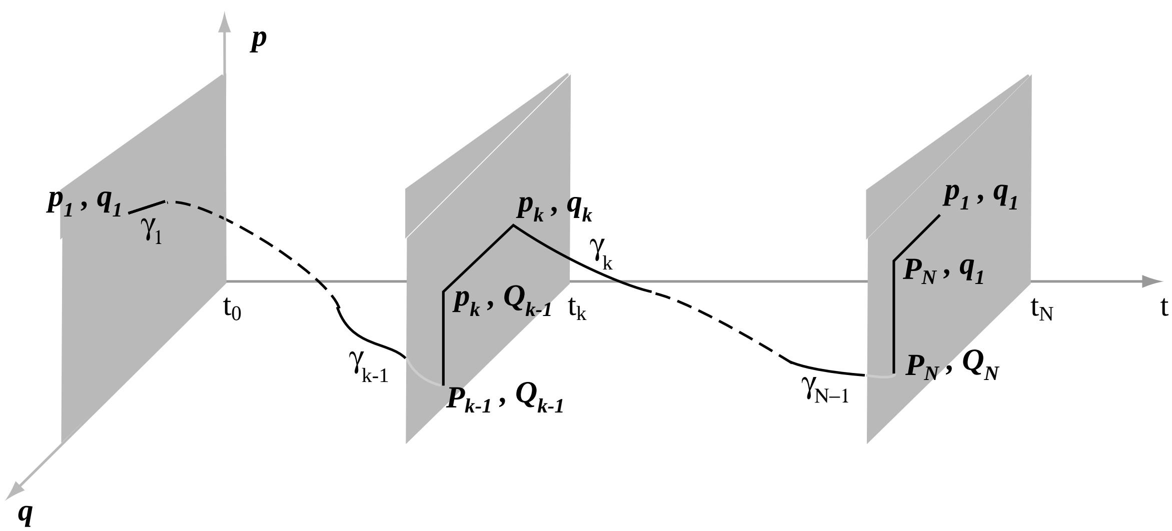

Fig. 52.1. Interpretation of $\tilde{W}$ as the action of a "broken" solution Γ, concatenation of the solution segments γ_k and "corners" in the $t = t_k$ planes.

Each curve γ_k in Figure 52.1 is the unique solution of Hamilton's equations starting at $(\boldsymbol{q}_k, \boldsymbol{p}_k, t_k)$ where $t_k = \frac{k-1}{N}$ and flowing for time $1/N$. Since $\tilde{S}_k(\boldsymbol{Q}_k, \boldsymbol{p}_k) = S_k(\boldsymbol{q}_k, \boldsymbol{p}_k) +$

$\boldsymbol{p}_k\boldsymbol{q}_k$ and $S_k(\boldsymbol{q}_k,\boldsymbol{p}_k)=\int_{\gamma_k}\boldsymbol{p}d\boldsymbol{q}-Hdt$ (see Theorem 59.7), $\tilde{W}$ measures the action of the broken solution Γ:

$$\begin{aligned}\tilde{W}(\boldsymbol{Q}_1,\boldsymbol{p}_1,\ldots,\boldsymbol{Q}_N,\boldsymbol{p}_N)&=\sum_{k=1}^N\boldsymbol{p}_k(\boldsymbol{q}_k-\boldsymbol{Q}_{k-1})+\sum_{k=1}^N\int_{\gamma_k}\boldsymbol{p}d\boldsymbol{q}-Hdt\\&=\int_\Gamma\boldsymbol{p}d\boldsymbol{q}-Hdt,\end{aligned}\tag{52.6}$$

where we have used the fact that, on the "corner" segments, $dt\equiv 0$, and on the vertical part of these corners, $d\boldsymbol{q}\equiv 0$. This is the definition given by Chaperon (1984) and (1989).

C. The Conley-Zehnder Theorem

The following theorem solved a famous conjecture by Arnold (1978) in the case of the torus. It was hailed as the start of symplectic topology, as it shows that symplectic diffeomorphisms have dynamics necessarily different from that of general diffeomorphisms, or even volume preserving diffeomorphisms. The original proof of Conley & Zehnder (1983) also reduces the analysis to finite dimensions, but by truncating Fourier series of periodic orbits. Chaperon's proof avoids the functional analysis altogether.

Theorem 52.4 (Conley-Zehnder) *Let h^1 be a Hamiltonian map of $\mathbb{T}^{2n}$. Then h^1 has at least $2n+1$ distinct fixed points and at least 2^n of them if they all are nondegenerate.*

Proof. Let $\tilde{W}$ be defined as in Proposition 52.3 for the decomposition of h^1 into symplectic maps close to Id given by (52.4). We will show that $\tilde{W}$ is equivalent to a g.p.q.i. on $\mathbb{T}^{2n}$, and hence, by Proposition 64.1, it has the prescribed number of critical points, corresponding to fixed points of h^1. We refer the reader to Section 64 for the definition and properties of generating phases that are relevant here. We first note that $\tilde{W}$ induces a function on $(\mathbb{R}^{2n})^N/\mathbb{Z}^{2n}$ where $\mathbb{Z}^{2n}$ acts on $(\mathbb{R}^{2n})^N$ by:

$$(\boldsymbol{m}_q,\boldsymbol{m}_p).(\boldsymbol{Q}_1,\boldsymbol{p}_1,\ldots,\boldsymbol{Q}_N,\boldsymbol{p}_N)=(\boldsymbol{Q}_1+\boldsymbol{m}_q,\boldsymbol{p}+\boldsymbol{m}_p,\ldots,\boldsymbol{Q}_N+\boldsymbol{m}_q,\boldsymbol{p}_N+\boldsymbol{m}_p)$$

The fact that $\tilde{W}$ is invariant under this action is most easily seen from (52.6). Indeed, since the Hamiltonian flow is a lift from one on $\mathbb{T}^{2n}$, the curve $\gamma_k+(\boldsymbol{m}_q,\boldsymbol{m}_p,0)$ is the solution between $(\boldsymbol{q}_k+\boldsymbol{m}_q,\boldsymbol{p}_k+\boldsymbol{m}_p)$ and $(\boldsymbol{Q}_k+\boldsymbol{m}_q,\boldsymbol{P}_k+\boldsymbol{m}_p)$ starting at time $\frac{k-1}{N}$ of that flow. But

$$\int_{\gamma_k+(\boldsymbol{m}_q,\boldsymbol{m}_p,0)} \boldsymbol{p}d\boldsymbol{q}+Hdt=\int_{\gamma_k}(\boldsymbol{p}+\boldsymbol{m}_p)d\boldsymbol{q}-Hdt=\boldsymbol{m}_p(\boldsymbol{Q}_k-\boldsymbol{q}_k)+\int_{\gamma_k}\boldsymbol{p}d\boldsymbol{q}-Hdt$$

Hence the action of γ_k changes by $\boldsymbol{m}_p(\boldsymbol{Q}_k-\boldsymbol{q}_k)$ under this transformation. On the other hand, under the same transformation, the sum $\sum_{k=1}^N \boldsymbol{p}_k(\boldsymbol{q}_k-\boldsymbol{Q}_{k-1})$ of Formula (52.6) changes by $\sum_{k=1}^N \boldsymbol{m}_p(\boldsymbol{q}_k-\boldsymbol{Q}_{k-1})$. Summing up the actions of the γ_k, these changes cancel out, from which we deduce that $\tilde{W}$ is invariant under the $\mathbb{Z}^{2n}$ action.

We now show that $\tilde{W}$ is equivalent to a g.p.q.i. over $\mathbb{T}^{2n}$. Let $E=(\mathbb{R}^{2n})^N\to\mathbb{R}^{2n}$ be the bundle given by the projection map onto $(\boldsymbol{Q}_N,\boldsymbol{p}_N)$ and let $\chi:E\to E$ be the bundle diffeomorphism given by:

$$\chi(\boldsymbol{Q}_1,\boldsymbol{p}_1,\dots,\boldsymbol{Q}_N,\boldsymbol{p}_N)=(\boldsymbol{a}_1,\boldsymbol{b}_1,\dots,\boldsymbol{a}_{N-1},\boldsymbol{b}_{N-1},\boldsymbol{Q}_N,\boldsymbol{p}_N)$$

where

$$\begin{aligned}\boldsymbol{a}_k&=\boldsymbol{Q}_k-\boldsymbol{Q}_{k-1}\qquad(\boldsymbol{Q}_0=\boldsymbol{Q}_N)\\ \boldsymbol{b}_k&=\boldsymbol{p}_k-\boldsymbol{p}_N.\end{aligned}$$

In these new coordinates, the $\mathbb{Z}^{2n}$ action only affects $(\boldsymbol{Q}_N,\boldsymbol{p}_N)$, so that $\tilde{W}\circ\chi^{-1}$ induces a function W on $(\mathbb{R}^{2n})^{N-1}\times\mathbb{T}^{2n}$. We now need to show that W is in fact a g.p.q.i. Define $\tilde{W}_0$ (resp. W_0) to be the functions $\tilde{W}$ (resp. W) obtained when setting the Hamiltonian to zero. Since $\tilde{S}_k(\boldsymbol{Q}_k,\boldsymbol{p}_k)=\boldsymbol{p}_k\boldsymbol{Q}_k$ in this case (see Remark 52.2), $\tilde{W}_0(\overline{\boldsymbol{Q}},\overline{\boldsymbol{p}})=\sum_{k=1}^N \boldsymbol{p}_k(\boldsymbol{Q}_k-\boldsymbol{Q}_{k-1})$ and hence a simple computation yields

$$W_0(\overline{\boldsymbol{a}},\overline{\boldsymbol{b}},\boldsymbol{Q}_N,\boldsymbol{p}_N)=\sum_{k=1}^{N-1}\boldsymbol{a}_k\cdot\boldsymbol{b}_k$$

which, as easily checked, is quadratic nondegenerate in the fiber. Finally, we show that $\frac{\partial}{\partial\boldsymbol{v}}(W-W_0)$ is bounded, where $\boldsymbol{v}=(\overline{\boldsymbol{a}},\overline{\boldsymbol{b}})$. It is sufficient for this to check that $d(\tilde{W}-\tilde{W}_0)$ is bounded. Using (52.5), we obtain:

$$\begin{aligned}d(\tilde{W}-\tilde{W}_0)=&\sum_{k=1}^N(\boldsymbol{P}_k-\boldsymbol{p}_{k+1})d\boldsymbol{Q}_k+\sum_{k=1}^N(\boldsymbol{q}_k-\boldsymbol{Q}_{k-1})d\boldsymbol{p}_k\\ &-\sum_{k=1}^N(\boldsymbol{Q}_k-\boldsymbol{Q}_{k-1})d\boldsymbol{p}_k-\sum_{k=1}^N(\boldsymbol{p}_k-\boldsymbol{p}_{k+1})d\boldsymbol{Q}_k\\ =&\sum_{k=1}^N(\boldsymbol{q}_k-\boldsymbol{Q}_k)d\boldsymbol{p}_k+\sum_{k=1}^N(\boldsymbol{P}_k-\boldsymbol{p}_k)d\boldsymbol{Q}_k\end{aligned}$$

where we have set throughout $\boldsymbol{Q}_0 = \boldsymbol{Q}_N, \boldsymbol{p}_{N+1} = \boldsymbol{p}_1$. Since by definition $(\boldsymbol{Q}_k, \boldsymbol{P}_k) = F_k(\boldsymbol{q}_k, \boldsymbol{p}_k)$ where $F_k = h_{\frac{k-1}{N}}^{\frac{k}{N}}$ lifts a diffeomorphism of $\mathbb{T}^{2n}$, the coefficients of the above differential must be bounded. We can conclude by applying Proposition 64.1. $\square$

Remark 52.5 Since the lift of the orbits we find are closed, the orbits in $\mathbb{T}^{2n}$ are contractible. In general, one cannot hope to find periodic orbits of different homotopy classes, as the example $H_0 \equiv 0$ shows. It would be interesting, however, to study the special properties of the set of rotation vectors that orbits of h^1 may have, i.e., to find out if being Hamiltonian implies more properties on this set than those known for general diffeomorphisms of $\mathbb{T}^{2n}$.

53. Generating Phases And Symplectic Geometry

We urge the reader to read Section 64, where we define generating phases as functions $W : E \to \mathbb{R}$, where E is a vector bundle over the manifold M. We then give conditions under which lower estimates on the number of critical points of W can be obtained from the topology of M. In this section, we show how such functions give rise to Lagrangian submanifolds of T^*M, hence the adjective "generating". In particular, we show that the action function obtained either in the symplectic twist map setting or in the Chaperon approach of last section generate a Lagrangian manifold canonically symplectomorphic to the graph of of the map F under consideration. More generally, this construction unifies the different finite, and even infinite, variational approaches in Hamiltonian dynamics.

A. Generating Phases and Lagrangian Manifolds

Let W be a differentiable function $M \to \mathbb{R}$. In Section 58.C, we show that:

$$dW(M) = \{(\boldsymbol{q}, dW(\boldsymbol{q})) \mid \boldsymbol{q} \in M\}$$

is a Lagrangian submanifold of T^*M. Note that this manifold is a graph over the zero section 0^*_M of T^*M. Heuristically, we would like to make it possible to similarly "generate" Lagrangian submanifolds that are not graphs with some kind of function. One way to do this is to add auxiliary variables and see our Lagrangian manifold as an appropriate projection in T^*M of a manifold in some bundle over M. This is what is behind the following construction.

Let $\pi : E \to M$ be a vector bundle over the manifold M. Let $W(\boldsymbol{q},\boldsymbol{v})$ be a real valued function on an open subset of E. The derivative $\frac{\partial W}{\partial \boldsymbol{v}} : E \to E^*$ of W along the fiber of E is well defined, in the sense that if U is a chart on M and $\psi_1, \psi_2 : U \times V \to \pi^{-1}(U)$ are two local trivializations of E, and $W_1 = W \circ \psi_1,\ W_2 = W \circ \psi_2$, then

$$\Phi^* \frac{\partial W_1}{\partial \boldsymbol{v}}(\boldsymbol{q},\boldsymbol{v}) d\boldsymbol{v} = \frac{\partial W_2}{\partial \boldsymbol{v}}(\Phi(\boldsymbol{q},\boldsymbol{v})) d\boldsymbol{v}$$

where $\Phi = \psi_2 \circ \psi_1^{-1}$ is the change of trivialization. We assume that the map: $(\boldsymbol{q},\boldsymbol{v}) \mapsto \frac{\partial W}{\partial \boldsymbol{v}}(\boldsymbol{q},\boldsymbol{v})$ is *transverse to* 0. This means that the second derivative (in any coordinates) $(\frac{\partial^2 W}{\partial \boldsymbol{v} \partial \boldsymbol{q}}, \frac{\partial^2 W}{\partial \boldsymbol{v}^2})$ is of maximum rank at points $(\boldsymbol{q},\boldsymbol{v})$ where $\frac{\partial W}{\partial \boldsymbol{v}}(\boldsymbol{q},\boldsymbol{v}) = 0$. With this assumption, the following set of *fiber critical points* is a manifold of same dimension as M:

$$\Sigma_W = \left\{ (\boldsymbol{q},\boldsymbol{v}) \in E \;\middle|\; \frac{\partial W}{\partial \boldsymbol{v}}(\boldsymbol{q},\boldsymbol{v}) = 0 \right\}. \tag{53.1}$$

[For a proof of this general fact about transversality, see *eg.* the theorem p.28 in Guillemin & Pollack (1974)]

Define the map:

$$\begin{aligned} i_W : \Sigma_W &\to T^*M \\ (\boldsymbol{q},\boldsymbol{v}) &\to \left(\boldsymbol{q}, \frac{\partial W}{\partial \boldsymbol{q}}(\boldsymbol{q},\boldsymbol{v}) \right) \end{aligned}$$

Exercise 53.1shows that this is an immersion. We now show directly that this immersion is Lagrangian:

$$i_W^* \boldsymbol{p} d\boldsymbol{q} = \frac{\partial W}{\partial \boldsymbol{q}}(\boldsymbol{q},\boldsymbol{v}) d\boldsymbol{q} = dW\big|_{\Sigma_W}(\boldsymbol{q},\boldsymbol{v})$$

and hence:

$$i_W^*(d\boldsymbol{q} \wedge d\boldsymbol{p}) = d^2 W\big|_{\Sigma_W} = 0.$$

We will say that W is a *generating phase* for a Lagrangian immersion $j : L \to T^*M$ if $j(L) = i_W(\Sigma_W)$.

Exercise 53.1 Show that $i_W : \Sigma_W \to T^*M$ is an immersion, *i.e.* that $Di_W\big|_{\Sigma_W}$ has full rank (*Hint.* Use the transversality condition to show that $Ker Di_W \cap T\Sigma_W = \{0\}$).

B. Symplectic Properties of Generating Phases

We start with the trivial, but important:

Proposition 53.2 *Suppose the Lagrangian submanifold $L \subset T^*M$ is generated by a function $W : E \to \mathbb{R}$. The points in the intersection of L with the zero section 0^*_M of T^*M are in a one to one correspondence with the critical points of W.*

Proof. $i_W(\boldsymbol{q}, \boldsymbol{v})$ is in L if and only if $\frac{\partial W}{\partial \boldsymbol{v}}(\boldsymbol{q}, \boldsymbol{v}) = 0$. It is in 0^*_M if and only if $\frac{\partial W}{\partial \boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{v}) = 0$. □

In Section 64, we find that critical points persist under elementary operations on generating phases: if $W_1 : E_1 \to \mathbb{R}$, and $W_2 : E_2 \to \mathbb{R}$ are two generating phases such that

$$W_2 \circ \Phi = W_1 + C,$$

or $W_2 : E_1 \times E_2 \to \mathbb{R}$ and

$$W_2(\boldsymbol{q}, \boldsymbol{v}_1, \boldsymbol{v}_2) = W_1(\boldsymbol{q}, \boldsymbol{v}_1) + f(\boldsymbol{q}, \boldsymbol{v}_2)$$

where Φ is a fiber preserving diffeomorphism, f is nondegenerate quadratic in $\boldsymbol{v}_2$ and C a constant, then W_1 and W_2 have the same number of critical points. The first operation is called equivalence, the second stabilization. This persistence is now geometrically explained by Proposition 53.2 and the following:

Lemma 53.3 *Two equivalent generating phases generate the same Lagrangian immersion. This is also true under stabilization.*

Proof. Let $W_2 \circ \Phi = W_1 + C$ where Φ is a fiber preserving diffeomorphism between $E_1 \to M$ and $E_2 \to M$. Writing $\Phi(\boldsymbol{q}, \boldsymbol{v}) = (\boldsymbol{q}, \phi(\boldsymbol{q}, \boldsymbol{v})) = (\boldsymbol{q}, \boldsymbol{v}')$, where $\boldsymbol{v} \to \phi(\boldsymbol{q}, \boldsymbol{v})$ is a diffeomorphism for each fixed $\boldsymbol{q}$, we have: $W_2(\boldsymbol{q}, \phi(\boldsymbol{q}, \boldsymbol{v})) = W_1(\boldsymbol{q}, \boldsymbol{v}) + C$. and thus

$$\frac{\partial W_1}{\partial \boldsymbol{v}} = \left(\frac{\partial W_2}{\partial \boldsymbol{v}}' \circ \Phi\right) . \frac{\partial \phi}{\partial \boldsymbol{v}}$$

This implies that $\Sigma_{W_2} = \Phi(\Sigma_{W_1})$, and we conclude the proof of the first assertion by noticing that:

$$\frac{\partial W_1}{\partial \boldsymbol{q}}(\boldsymbol{q},\boldsymbol{v}) = \frac{\partial W_2}{\partial \boldsymbol{q}}(\Phi(\boldsymbol{q},\boldsymbol{v})).$$

Now let $W_2(\boldsymbol{q},\boldsymbol{v}_1,\boldsymbol{v}_2) = W_1(\boldsymbol{q},\boldsymbol{v}_1) + f(\boldsymbol{q},\boldsymbol{v}_2)$ where f is quadratic and nondegenerate in $\boldsymbol{v}_2$. We have:

$$\partial W_2/\partial \boldsymbol{v} = 0 \Leftrightarrow \boldsymbol{v}_2 = 0 \quad \text{and} \quad \partial W_1/\partial \boldsymbol{v}_1 = 0$$

so that $\Sigma_{W_2} = \Sigma_{W_1} \times 0_{E_2}$, where 0_{E_2} is the zero section of E_2. Moreover $\partial f/\partial \boldsymbol{q}\big|_{\{\boldsymbol{v}_2=0\}} = 0$ so that, at points $(\boldsymbol{q},\boldsymbol{v}_1,0)$ of Σ_2,

$$\left(\boldsymbol{q}, \frac{\partial W_2}{\partial \boldsymbol{q}}(\boldsymbol{q},\boldsymbol{v}_1,0)\right) = \left(\boldsymbol{q}, \frac{\partial W_1}{\partial \boldsymbol{q}}(\boldsymbol{q},\boldsymbol{v}_1)\right).$$

$\square$

C. The Action Function Generates the Graph of F

We examine here the twist map case, and let the reader perform the analysis for the Chaperon case in Exercise 53.4. Let M be an n–dimensional manifold and F be a symplectic twist map on $U \subset T^*M$, where U is of the form $\{(\boldsymbol{q},\boldsymbol{p}) \in T^*M \mid \|\boldsymbol{p}\| < K\}$. Let $S(\boldsymbol{q},\boldsymbol{Q})$ be a generating function for F. S can be seen as a function on some open set V of $M \times M$, diffeomorphic to U. [19] Since $\boldsymbol{P}d\boldsymbol{Q} - \boldsymbol{p}d\boldsymbol{q} = dS(\boldsymbol{q},\boldsymbol{Q})$, we can describe the graph of F as:

$$Graph(F) = \left\{\left(\boldsymbol{q}, -\frac{\partial S}{\partial \boldsymbol{q}}(\boldsymbol{q},\boldsymbol{Q}), \boldsymbol{Q}, \frac{\partial S}{\partial \boldsymbol{Q}}(\boldsymbol{q},\boldsymbol{Q})\right) \;\Big|\; (\boldsymbol{q},\boldsymbol{Q}) \in V\right\} \subset (T^*M)^2,$$

which is canonically symplectomorphic to (see the map j below):

$$\left\{\left(\boldsymbol{q}, \boldsymbol{Q}, \frac{\partial S}{\partial \boldsymbol{q}}(\boldsymbol{q},\boldsymbol{Q}), \frac{\partial S}{\partial \boldsymbol{Q}}(\boldsymbol{q},\boldsymbol{Q})\right) \;\Big|\; (\boldsymbol{q},\boldsymbol{Q}) \in V\right\} \subset T^*(M \times M).$$

One can easily check that this manifold has S as a generating phase. In other words *the generating function of a symplectic twist map F is a generating phase for the graph of F*. Consider now the more general case where $F = F_N \circ \ldots \circ F_1$ is a product of symplectic twist maps of $U \subset T^*M$. This time, the candidate for a generating phase is:

$$W(\overline{\boldsymbol{q}}) = \sum_{k=1}^{N} S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}),$$

[19] in the case where $M = \mathbb{T}^n$, and the map is defined on all of $T^*\mathbb{T}^n$, we have $V \cong \tilde{U} \cong \mathbb{R}^{2n}$.

where we do not identify $\boldsymbol{q}_{N+1}$ *and* $\boldsymbol{q}_1$ *in any way*. Then, writing

$$\boldsymbol{v} = (\boldsymbol{q}_2, \dots, \boldsymbol{q}_N), \quad \boldsymbol{q} = (\boldsymbol{q}_1, \boldsymbol{q}_{N+1}),$$

we show that $W(\boldsymbol{q}, \boldsymbol{v})$ is a generating phase for $Graph(F) \subset (T^*M)^2$. Let

$$\mathcal{U} = \big\{(\boldsymbol{q}_1, \dots, \boldsymbol{q}_{N+1}) \in M^{N+1} \;\mid\; (\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) \in \psi_k(U)\big\}$$

where ψ_k is the "Legendre transformation" attached to the twist map F_k. Let $\beta : M^{N+1} \to M \times M$ be the map defined by: $(\boldsymbol{q}_1, \dots, \boldsymbol{q}_{N+1}) \to (\boldsymbol{q}_1, \boldsymbol{q}_{N+1})$. The bundle that we will consider here is:

$$\mathcal{U} \to \beta(\mathcal{U}) \subset M \times M.$$

The Critical Action Principle (Proposition 23.2, and Exercise 26.4) states that $\frac{\partial W}{\partial \boldsymbol{v}}(\boldsymbol{q}, \boldsymbol{v}) = 0$ exactly when $\overline{\boldsymbol{q}} = (\boldsymbol{q}, \boldsymbol{v})$ is the $\boldsymbol{q}$ component of the orbit of $(\boldsymbol{q}_1, \boldsymbol{p}_1(\boldsymbol{q}_1, \boldsymbol{q}_2))$ under the successive F_k's. This means that *the set of orbits under the successive F_k's is in bijection with the set $\Sigma_W = \{\frac{\partial W}{\partial \boldsymbol{v}}(\boldsymbol{q}, \boldsymbol{v}) = 0\}$ as defined in (53.1)* . Since this set is parameterized by the starting point of an orbit, it is diffeomorphic to U, hence a manifold. For $\overline{\boldsymbol{q}} \in \Sigma_W$, we have:

$$F(\boldsymbol{q}_1, \boldsymbol{p}_1(\boldsymbol{q}_1, \boldsymbol{q}_2)) = \big(\boldsymbol{q}_{N+1}, \boldsymbol{P}_{N+1}(\boldsymbol{q}_N, \boldsymbol{q}_{N+1})\big)$$

but:

$$\boldsymbol{p}_1(\boldsymbol{q}_1, \boldsymbol{q}_2) = -\partial_1 S_1(\boldsymbol{q}_1, \boldsymbol{q}_2) = -\frac{\partial W}{\partial \boldsymbol{q}_1}(\boldsymbol{q}_1, \boldsymbol{q}_{N+1}, \boldsymbol{v})$$
$$\boldsymbol{P}_{N+1}(\boldsymbol{q}_N, \boldsymbol{q}_{N+1}) = \partial_2 S_N(\boldsymbol{q}_N, \boldsymbol{q}_{N+1}) = \frac{\partial W}{\partial \boldsymbol{q}_{N+1}}(\boldsymbol{q}_1, \boldsymbol{q}_{N+1}, \boldsymbol{v})$$

In other words, the graph of F in $T^*M \times T^*M$ can be expressed as:

$$Graph(F) = \left\{ \left(\boldsymbol{q}_1, -\frac{\partial W}{\partial \boldsymbol{q}_1}(\boldsymbol{q}, \boldsymbol{v}), \boldsymbol{q}_{N+1}, \frac{\partial W}{\partial \boldsymbol{q}_{N+1}}(\boldsymbol{q}, \boldsymbol{v})\right) \;\middle|\; (\boldsymbol{q}, \boldsymbol{v}) \in \Sigma_W \right\}.$$

To finish our construction, we define the following symplectic map:

$$j : (T^*M \times T^*M, -\Omega_M \oplus \Omega_M) \to (T^*(M \times M), \Omega_{M \times M})$$
$$(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{Q}, \boldsymbol{P}) \to (\boldsymbol{q}, \boldsymbol{Q}, -\boldsymbol{p}, \boldsymbol{P}).$$

where Ω_X denotes the canonical symplectic structure on T^*X. Clearly:

$$j(Graph(F)) = i_W(\Sigma_W),$$

that is, W generates the Lagrangian manifold $Graph(F)$. Note that the fixed points of F correspond to $Graph(F) \cap \Delta(T^*M \times T^*M)$, i.e. to $\overline{\boldsymbol{q}} \in \Sigma_W$ such that $\boldsymbol{q}_1 = \boldsymbol{q}_{N+1}$ and $-\partial_1 S_1(\boldsymbol{q}_1, \boldsymbol{q}_2) = \partial_2 S_N(\boldsymbol{q}_N, \boldsymbol{q}_{N+1})$, which are critical points of $W\big|_{\{\boldsymbol{q}_1 = \boldsymbol{q}_{N+1}\}}$, as we well know.

Exercise 53.4 Show that the generating function W of Chaperon (see Proposition 52.3) generates the graph of the Hamiltonian map $F : \mathbb{T}^{2n} \to \mathbb{T}^{2n}$. (*Hint.* If you are stuck, consult Laudenbach & Sikorav (1985)).

D. Symplectic Reduction

We introduce yet another geometric point of view for the generating phase construction. We will see that if a Lagrangian manifold $L \subset T^*M$ is generated by the phase $W : E \to \mathbb{R}$, than in fact L is the symplectic reduction of the Lagrangian manifold $dW(E) \subset T^*E$. We introduce symplectic reduction in the linear case, and only sketch briefly the manifold case, referring the reader to Weinstein (1979) for more detail.

Consider a symplectic vector space V, Ω_0 of dimension $2n$. Let C be a coisotropic subspace of V. Let $\Lambda(V)$ be the set of Lagrangian subspaces of V (a Grassmanian manifold). The process of symplectic reduction gives a natural map $\Lambda(V) \to \Lambda\left(C/C^\perp\right)$ that we now describe. By Theorem 55.1, we know that we can find symplectic coordinates for V in which:

$$C = \{(q_1, \ldots, q_n, p_1, \ldots, p_k)\}$$

and we have $C^\perp = \{(q_{k+1}, \ldots, q_n)\} \subset C$. Then

$$C/C^\perp \simeq \{(q_1, \ldots, q_k, p_1, \ldots, p_k)\}$$

which is obviously symplectic. It is called the *reduced symplectic space* along C. We denote by Red the quotient map $C \to C/C^\perp$. The symplectic form Ω_C of $C/C^\perp$ is natural in the sense that it makes Red into a symplectic map:

$$\Omega_C(Red(\boldsymbol{v}), Red(\boldsymbol{v}')) = \Omega(\boldsymbol{v}, \boldsymbol{v}'). \tag{53.2}$$

Proposition 53.5 *Let $L \subset V$ be a Lagrangian subspace and $C \subset V$ a coisotropic subspace. Then*

$$L_C = Red(L \cap C) = L \cap C / L \cap C^{\perp}$$

is Lagrangian in $C/C^{\perp}$.

We say that L_C is *the symplectic reduction* of L along the coisotropic space C.

Proof. Formula (53.2) tells us that L_C is isotropic. We need to show that $dimL_C = \frac{1}{2}dimC/C^{\perp}$. From linear algebra:

$$dimL_C = dim(L \cap C) - dim(L \cap C^{\perp}).$$

As would be the case with any nondegenerate bilinear form, the dimensions of a subspace and that of its orthogonal add up to the dimension of the ambient space. Also, the orthogonal of an intersection is the sum of the orthogonal. Hence:

$$dimV = dim(L \cap C^{\perp}) + dim(L \cap C^{\perp})^{\perp} = dim(L \cap C^{\perp}) + dim(L + C),$$

since $L^{\perp} = L$. Thus

$$\begin{aligned} dimL_C &= dim(L \cap C) - dimV + dim(L + C) = dimL + dimC - dimV \\ &= dimC - \frac{1}{2}dimV. \end{aligned} \tag{53.3}$$

On the other hand:

$$\begin{aligned} dim(C/C^{\perp}) &= dimC - dimC^{\perp} = dimC - (dimV - dimC) \\ &= 2dimC - dimV \end{aligned} \tag{53.4}$$

We conclude that $dimL_C = \frac{1}{2}dim(C/C^{\perp})$ by putting (53.3) and (53.4) together. □

We now sketch the reduction construction in the manifold case. Let C be a coisotropic submanifold of a symplectic manifold (M, Ω). Then $TC^{\perp}$ is a subbundle of TC (that is, the fibers are of same dimension and vary smoothly) so we can form the quotient bundle $TC/TC^{\perp}$, with base C and fiber the quotient $T_{\boldsymbol{q}}C/T_{\boldsymbol{q}}C^{\perp}$ at each point $\boldsymbol{q}$ of C. It turns out that this quotient bundle can actually be seen as the tangent bundle of a certain manifold $C/C^{\perp}$, whose points are leaves of the integrable foliation $TC^{\perp}$. Moreover one can show that the naturally induced form Ω_C is indeed symplectic on $C/C^{\perp}$. Finally, we define $red : C \to C/C^{\perp}$ as the projection. Its derivative is basically the map Red defined above. One can show that, if C intersect a Lagrangian submanifold L transversally, then $L_C = red(L)$ is an immersed symplectic manifold of $C/C^{\perp}$, which is the *reduction of L along C*.

We now apply this new point of view to the generating function construction. Let $E = M \times \mathbb{R}^N$. We show that if $L = i_W(\Sigma_W) \subset T^*M$ is generated by the generating phase $W : E \to \mathbb{R}$, then L is in fact the reduction of $dW(E) \subset T^*E$ along the coisotropic manifold $C = \{\boldsymbol{p_v} = 0\}$, where we have given T^*E the coordinate $(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{p_q}, \boldsymbol{p_v})$. This is just a matter of checking through the construction. We know that $dW(E)$ is Lagrangian in T^*E. Its intersection with C is the set:

$$\begin{aligned} dW(E) \cap C =& \left\{ (\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{p_q}, \boldsymbol{p_v}) \in T^*E \;\middle|\; \boldsymbol{p_q} = \frac{\partial W}{\partial \boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{v}),\ \boldsymbol{p_v} = \frac{\partial W}{\partial \boldsymbol{v}}(\boldsymbol{q}, \boldsymbol{v}) = 0 \right\} \\ =& dW(\Sigma_W). \end{aligned}$$

where Σ_W is the set of fiber critical points in E. Since by the transversality condition in our definition of generating phase Σ_W is a manifold, so is $dW(E) \cap C$: for any W, the map $dW : E \to T^*E$ is an embedding. The bundle $TC^{\perp}$ is the one generated by the vector fields $\frac{\partial}{\partial \boldsymbol{v}}$ and thus $C/C^{\perp}$ can be identified with $T^*M = \{(\boldsymbol{q}, \boldsymbol{p_q})\}$. The image of $dW(E) \cap C$ under the projection $red : C \to C/C^{\perp}$ is exactly $i_W(\Sigma_W) = \{(\boldsymbol{q}, \frac{\partial W}{\partial \boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{v})) \mid \frac{\partial W}{\partial \boldsymbol{v}}(\boldsymbol{q}, \boldsymbol{v}) = 0\} = L$. Note that because $E = M \times \mathbb{R}^N$, the above argument is independent of the coordinate chosen (*eg.* C is well defined). With a little care, the argument extends to the case where E is a nontrivial bundle over M.

Exercise 53.6 Show that, in the Darboux coordinate used above, the $\boldsymbol{q}$–plane and the $\boldsymbol{p}$–plane of V both reduce to the $\boldsymbol{q}$ and $\boldsymbol{p}$–plane (resp.) of $C/C^{\perp}$.

E. Further Applications Of Generating Phases

The symplectic theory of generating phases does not only provide a unifying packaging for the different variational approaches to Hamiltonian systems. It can also serve as the basis of symplectic topology, where invariants called *symplectic capacities* play a crucial role. Roughly speaking, capacities are to symplectic geometry what volume is to Riemannian geometry: they provide obstructions for sets to be symplectomorphic, or for sets to be symplectically squeezed inside other sets. Viterbo (1992) uses generating phases to define such capacities, in contrast to prior approaches by Gromov (1985) who uses the theory of pseudo–holomorphic curves. The basis of the definition of capacity in Viterbo (1992) is a converse statement to Lemma 53.3:

Proposition 53.7 *If W_1 and W_2 both generate $h^t(0^*_M)$, where h^t is a Hamiltonian isotopy, then after stabilization W_1 and W_2 are equivalent.*

In view of this, Viterbo is able to define a capacity for a Lagrangian manifold L Hamiltonian isotopic to 0^*_M by choosing minimax values of a given (and hence any) generating phase for L.

In another work, Viterbo (1987) shows that a certain integer function called Maslov Index on the set of paths in the Lagrangian Grassmannian is invariant under symplectic reduction. It can be shown that the Lagrangian Grassmanian $\Lambda(V)$ has first fundamental group $\pi_1(\Lambda(V)) = \mathbb{Z}$. As mentioned in Chapter 6, we can roughly interpret this by saying that $\Lambda(V)$ has a "hole" and the *Maslov index* measures the number of turns a curve makes around that hole. Now let W_t be the generating phases for a Hamiltonian isotopy h^t. The set $dW_t(E)$ is Lagrangian in T^*E and its reduction is the graph of h^t (where W_t is the action function for a decomposition of h^t). The Maslov Index in $\Lambda(T^*E)$ detects the change in Morse Index of the second derivative of W_t, whereas on the graph of h^t, it detects a non transverse intersection with the plane $\{(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P})\}$. This can be used to give a neat generalization to Lemma 29.4 and to explain the classical relationship discovered by Morse between the index of the second variation of the action function and the number of conjugate points (see Milnor (1969) for the classical, Riemannian geometry case, and Duistermaat (1976) for the more general convex Lagrangian case).