1

TWIST MAPS OF THE ANNULUS

4. Monotone Twist Maps

A. Definitions

The *annulus* can be defined as

$$\mathbf{A} = \mathbb{S}^1 \times [a, b],$$

where the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. We define the *cylinder* by:

$$\mathcal{C} = \mathbb{S}^1 \times \mathbb{R}.$$

As with maps of the circle, it is often less ambiguous to work with lifts of diffeomorphisms of $\mathbf{A}$. These are maps of the *strip*:

$$\mathcal{A} := \{(x, y) \in \mathbb{R}^2 \mid a \leq y \leq b\}$$

where x, thought of as the angular variable, ranges over $\mathbb{R}$ (one can also consider more general strips of the type $\{(x, y) \mid \ u_-(x) \leq y \leq u_+(x)\}$ for two differentiable functions u_-, u_+. The results are identical). The *covering map* $proj : \mathcal{A} \mapsto \mathbf{A}$ takes (x, y) to $(x \mod 1\ , y)$ and a *lift* of a map f of the annulus is a map F of the strip which satisfies:

$$proj \circ F = f \circ proj.$$

This implies in particular that $F(x + 1, y) = F(x, y) + (n, 0)$, for some integer n. By continuity, n does not depend on the point (x, y), nor on the lift F of f, it is called the *degree* of f. In this book, we assume that f is an orientation preserving diffeomorphism of the annulus. In this case, the degree of f is 1 and

$$F(x+1,y) = F(x,y) + (1,0) \tag{4.1}$$

for any lift F of f. Denoting by T the translation $T(x,y) = (x+1,y)$, Equation (4.1) reads:

$$F \circ T = T \circ F \tag{4.2}$$

Clearly, any map F of $\mathcal{A}$ which satisfies (4.2) is the lift of a map f of $\mathbf{A}$ of degree 1. We say that f is *induced* by F.

Definition 4.1 Let F be a diffeomorphism of $\mathcal{A} = \mathbb{R} \times [a,b]$ and write $(X(x,y), Y(x,y)) = F(x,y)$. Let F satisfy:

(1) F *preserves the boundaries of* $\mathcal{A}$: $Y(x,a) = a, Y(x,b) = b$.

(2) *Twist Condition:* the function $y \mapsto X(x_0, y)$ is strictly monotone for each given x_0.

(3) *Area and Orientation Preserving:* $\det DF = 1$ or, equivalently, $dY \wedge dX = dy \wedge dx$.

(4) $F \circ T = T \circ F$

Then F induces a map f on the annulus $\mathbf{A}$ which is called a *(area preserving, monotone) twist map of the annulus*.

Exercise 4.2 Prove the above statements about the degree of a map and its lifts.

B. Comments on the Definition

Twist Condition.

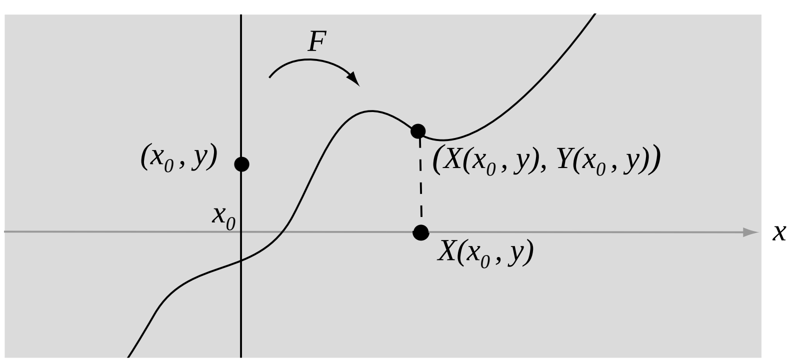

Fig. 4.0. The positive twist condition: as one moves up along a vertical fiber, the image point moves right.

Condition (2) implies that the map $y \mapsto X(x_0, y)$ is a diffeomorphism between the vertical *fiber* $\{x = x_0\}$ and its image on the x-axis (also called the *base*). In other words, the image of the fiber $\{x = x_0\}$ by F forms a graph over the x-axis, as is shown in Figure 4.1.

We say that F is a *positive twist map* (resp. *negative twist map*) if $y \mapsto X(x_0, y)$ is strictly increasing (resp. decreasing). Since in this book we consider differentiable maps, we can express the monotonicity of the map $y \mapsto X(x_0, y)$ by the equivalent derivative condition:

$$\frac{\partial X}{\partial y} \neq 0. \tag{4.3}$$

Since $\mathcal{A}$ is connected, this derivative is either always strictly positive (giving a positive twist map) or always strictly negative (for negative twist maps). Note that the lift of a positive twist map “moves” points on the upper boundary of $\mathcal{A}$ “faster” than on the lower boundary. If F satisfies just the latter property, we say that it has the *boundary twist condition.* This condition, much weaker than the twist condition in Definition 4.2 is all that is needed in the Poincaré-Birkhoff theorem, see Section 7.

We now show that the twist condition implies that the map $\psi : (x, y) \mapsto (x, X)$ is an *embedding* of $\mathcal{A}$ in $\mathbb{R}^2$, *i.e.* a local diffeomorphism which is injective. Indeed, the differential of ψ is given by :

$$D\psi = \begin{pmatrix} 1 & 0 \\ \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \end{pmatrix}$$

whose determinant $\frac{\partial X}{\partial y}$ is non zero by the twist condition. Hence ψ is a local diffeomorphism. To show that it is injective, suppose $\psi(x_1, y_1) = \psi(x_2, y_2)$. Then, trivially, $x_1 = x_2$, and y_1 and y_2 are forced to be equal because the map $y \mapsto X(x_1, y)$ is strictly monotone. We leave it to the reader to verify that, conversely, if ψ is an embedding of $\mathcal{A}$, then the twist condition is satisfied. We treat ψ as a change of coordinates. We will sometimes use the notation Ψ_F for ψ, to emphasize its dependence on the map F.

Area Preservation, Flux and Symplecticity. The change of variable formula in multivariate integration shows that the infinitesimal condition $\det DF = 1$ implies $Area(X) = Area(F(X))$ for any domain X in $\mathcal{A}$ (or for any Lebesgue measurable set X). We now relate area preservation to another global notion: that of flux. For an area preserving map F of $\mathcal{A}$, define the function $S : \mathcal{A} \to \mathbb{R}$ by:

$$S(z) = \int_{z_0}^{z} Y dX - y dx$$

where this path integral is over *any* curve joining a chosen base point z_0 and the variable $z = (x, y)$. Using Condition (3), Stokes' theorem and the fact that $\mathcal{A}$ is simply connected, one shows that S is well defined on $\mathcal{A}$ ($i.e.$ it is independent of the path of integration) and that $Y dX - y dx = dS$ (see Exercise 4.3). The *flux* of an area preserving map F of $\mathbb{R}^2$ satisfying $F \circ T = T \circ F$ is defined by:

$$Flux(F) = S(Tz) - S(z)$$

Note that this makes sense, since, by Stokes' theorem, $S \circ T - S$ is constant (see Exercise 4.4). The flux of the map F can be seen geometrically in the cylinder $\mathcal{C}$ as the net area comprised between an embedded circle wrapping once around $\mathcal{C}$ and its image by the map f induced by F (see Figure 4.2). Indeed, $S(\beta(1)) - S(\beta(0)) = \int_\beta Y dX - y dx = \int_{F(\beta)} y dx - \int_\beta y dx$ for any curve $\beta : [0, 1] \mapsto \mathcal{A}$. Now take β such that $\beta(1) = T\beta(0)$.

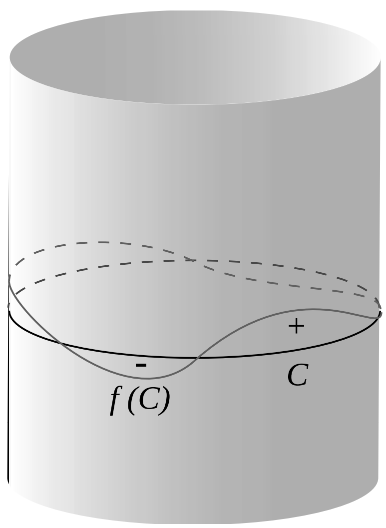

Fig. 4.2. The flux of a cylinder map as the net area between an enclosing circle C and its image $f(C)$

If F preserves the boundary of a bounded strip $\mathcal{A}$, then f preserves the boundary circles and the flux is by force zero. When no such curve is preserved for maps on the cylinder, the flux can take any value in $\mathbb{R}$ as the example $V_a(x, y) = (x, y + a)$ with $Flux(F) = a$ shows. Since examples of this type show no recurrent dynamics, we exclude them from our study by always imposing, directly or indirectly, the zero flux condition on our maps. If F has zero flux, then $S \circ T = S$ and thus S induces a function s on $\mathbf{A}$ such that

$$f^*(ydx) - ydx = ds. \tag{4.4}$$

taking the exterior derivative on both sides of this equation, one gets $d(f^*ydx - ydx) = d^2s = 0$, and thus

$$f^*(dx \wedge dy) = dx \wedge dy. \tag{4.5}$$

A map that satisfies this last equality is called *symplectic*, because it preserves the *symplectic form* $dx \wedge dy$. In the present, 2 dimensional case the symplectic for is just the area form (see Chapter 4 and Appendix 1 for generalizations to higher dimensions). A map f that satisfies (4.4) is called *exact symplectic*. Hence (4.5) shows that exact symplectic implies symplectic. We have shown that if F has zero flux, the map f it induces is exact symplectic. Conversely, by Stokes' theorem, if f is exact symplectic, any of its lifts has zero flux (Exercise 4.3). Hence the map V_a of the cylinder defined above is not exact symplectic, even though it is symplectic. Note that, in contrast, a symplectic map F of the plane is always exact symplectic: as any closed form on the plane, $F^*(ydx) - ydx$ is exact (Poincaré's Lemma).

Exercise 4.4 a) Using Stokes Theorem, show that if λ is a closed 1–form on a simply connected domain of $\mathbb{R}^2$, then the function $S = \int_{z_0}^{z} \lambda$ is well defined (*i.e.* does not depend on the path of integration between z and z_0) and that $dS = \lambda$. Apply this to $\lambda = YdX - ydx$.
b) What should a definition of S be if F preserves a smooth area form $\alpha(x,y)dy \wedge dx$?

Exercise 4.5 a) Let F be an area preserving map of $\mathbb{R}^2$ with $F \circ T = T \circ F$. Show that for the function S defined above, $S \circ T - S$ is constant, and hence $Flux(F)$ is well defined. (*Hint.* Given two points z_1, z_2 in $\mathcal{A}$, take any two curves γ_1, γ_2, with γ_i joining z_i and $Tz_i, i = 1, 2$. Take a curve β joining z_1 and z_2 and apply Stokes Theorem to the closed curve $\beta \cdot \gamma_1 \cdot (T\beta)^{-1} \cdot \gamma_2^{-1}$ and the form $YdX - ydx$.)
b) Show that any lift of an exact symplectic map of the cylinder has zero flux.
c) (For those who know some DeRham cohomology) Prove that $Flux(F)$ is the result of the pairing of the class $[f^*ydx - ydx]$ in $H^1_{DR}(\mathcal{C})$ with the first homology class represented by a circle going around the cylinder once in the positive direction (as usual, f is the map induced by F).

C. Twist Maps of the Cylinder

The comments of the previous subsection motivate the following:

Definition 4.6 (Twist Maps of the Cylinder) Let F be a diffeomorphism of $\mathbb{R}^2$ and write $(X(x,y), Y(x,y)) = F(x,y)$. Let F satisfy:

(1) *F is isotopic to the Identity*

(2) *Twist Condition:* the map $\psi := (x,y) \mapsto (x, X(x,y))$ is a diffeomorphism of $\mathbb{R}^2$

(3) *Area Preserving & Zero Flux (Exact Symplectic):* $YdX - ydx = dS$ for some real valued function S on $\mathbb{R}^2$ satisfying:

$$S(x+1, y) = S(x,y).$$

Then F is the lift of a map f on the cylinder $\mathcal{C}$ which is called a *monotone twist map of the cylinder.*

Condition (1) means that F can be deformed continuously into the identity through a path of homeomorphisms of the cylinder. For maps of the closed strip $\mathbb{R} \times [a,b]$, this condition clearly implies that the boundaries have to be preserved, and hence Condition (1) here is the analog to Condition (1) in Definition 4.1. It will appear clearly in the next section that the periodicity of the function S implies the periodicity $F \circ T = T \circ F$, *i.e.* Condition (4) of Definition 4.1, which is necessary for F to induce a map of the cylinder. In turn, the periodicity condition $F \circ T = T \circ F$ implies that F is homotopic to Id. Finally, the condition that ψ be a diffeomorphism here can be relaxed: one can require that ψ only be an embedding, *i.e.* a diffeomorphism of $\mathbb{R}^2$ into a proper subset of $\mathbb{R}^2$, to the cost of some (manageable) complications in the development of the theory.

Remark 4.7 There exist several other definitions of monotone twist maps in the literature. Most noteworthy are the topological definitions, where the map is only required to be a homeomorphism (and not necessarily a diffeomorphism). The twist condition takes different forms with different authors. One commonly used is that the map $y \mapsto X(x,y)$ be monotonic (Boyland (1988), Hall (1984), Katok (1982), LeCalvez (1991)). A much milder condition is considered in Franks (1988), where certain neighborhoods must move in opposite directions around the annulus. The preservation of area is sometimes discarded by these authors, replaced by a condition that the map contracts the area, or that it is topologically recurrent. The topological theory for twist maps is extremely rich and would be the subject of an entire book. Our choice of working in the differentiable category stems from the possibilities of generalization to higher dimensions that it offers.

Exercise 4.8 Show that a map of the bounded annulus which is homotopic to Id preserves each boundary component (Note: the converse is also true, but much harder to prove).

5. Generating Functions and the Variational Setting

A. Generating Functions

In the previous section, we have seen that the lift F of a twist map of either the cylinder or the annulus comes with a function S such that $F^*ydx - ydx = YdX - ydx = dS$ and $S(x+1,y) = S(x,y)$. The first equation expresses the fact that F preserves the area, whereas the periodicity of S, expresses the zero flux condition.

On the other hand, the twist condition on F gives us a function ψ which we view as a change of coordinates $\psi : (x,y) \mapsto (x,X)$. In the (x,X) coordinates[5] the equation $YdX - ydx = dS(x,X)$ implies immediately that the functions $-y(x,X)$ and $Y(x,X)$ are the partial derivatives of S:

$$y = -\frac{\partial S(x,X)}{\partial x}, \qquad Y = \frac{\partial S(x,X)}{\partial X} \tag{5.1}$$

These simple equations are the cornerstone of this book. The function $S(x,X)$ is called the *generating function* of F in that from S we can retrieve F, at least implicitly: ψ^{-1} is given by $(x,X) \mapsto (x, -\frac{\partial S}{\partial x})$ hence ψ is implicitly given by S. Thus F, which can be defined by:

$$F : (x,y) \mapsto (X \circ \psi(x,y), \frac{\partial S}{\partial X}(\psi(x,y))) \tag{5.2}$$

is also implicitly given by the function S and its partial derivatives. In Proposition 25.2 of Chapter 4, we give conditions under which a function on $\mathbb{R}^2$ is a generating function of

[5] Remember that under the change of coordinates ψ, a function S changes according to $S \mapsto S \circ \psi$. Likewise, $y \mapsto y \circ \psi$ and $Y \mapsto Y \circ \psi$.

the lift F of some twist map. In 25.1 we also show that the correspondence between maps and their generating functions (mod constant) is one to one and continuous. The following exercise gives two necessary conditions for a function to generate a twist map.

Exercise 5.1 Show that if $S(x, X)$ is the generating function of a positive twist map, then:
a) $\partial_{12} S(x, X) < 0$
b) $S(x + 1, X + 1) = S(x, X)$
Note that, in the litterature, the condition in a) is often taken as the twist condition

Exercise 5.2 Show that if F is the lift of a twist map of the annulus $\mathbb{S}^1 \times [0, 1]$ then $S(x, X)$ can be interpreted as the area of the triangular shaped area with vertices $(x, 0), (X, 0)$ and (X, Y) shown in Figure 5.1. (*Hint.* Show geometrically on this picture that $Y = \frac{\partial S}{\partial X}$. For $y = -\frac{\partial S}{\partial x}$, consider the preimage of this triangular region by F). Solve question b) of the previous exercise using this geometric construction.

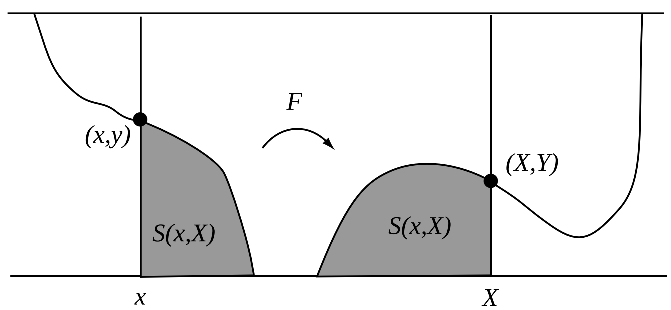

Fig. 5.1. The generating function as an area

Exercise 5.3 Show that the inverse of a positive twist map with generating function $S(x, X)$ is a negative twist map with generating function $-S(X, x)$.

B. Variational Principle

The lift F of a twist map gives rise to a dynamical system whose orbits are given by the images of points of $\mathbb{R}^2$ under the successive iterates of F. The *orbit* of the point (x_0, y_0) is the biinfinite sequence:

$$\{\ldots (x_{-1}, y_{-1}), (x_0, y_0), (x_1, y_1), \ldots, (x_k, y_k) \ldots\}$$

where $(x_k, y_k) = F(x_{k-1}, y_{k-1})$.

Lemma 5.4 *Let F be the lift of a monotone twist map of $\mathcal{A}$ or $\mathbb{R}^2$ and let $S(x, X)$ be its generating function. There is a one to one correspondence between orbits $\{(x_k, y_k) = F^k(x_0, y_0)\}_{k\in\mathbb{Z}}$ of F and sequences $\{x_k\}_{k\in\mathbb{Z}}$ satisfying:*

$$\partial_1 S(x_k, x_{k+1}) + \partial_2 S(x_{k-1}, x_k) = 0 \qquad \forall k \in \mathbb{Z}. \tag{5.3}$$

The correspondence is given by: $y_k = -\partial_1 S(x_k, x_{k+1})$.

Proof. Let $\{(x_k, y_k)\}_{k\in\mathbb{Z}}$ be an orbit of F. Since $(x_k, y_k) = F(x_{k-1}, y_{k-1})$ for all integer k, Equation (5.1) implies:

$$y_k = -\partial_1 S(x_k, x_{k+1}) = \partial_2 S(x_{k-1}, x_k).$$

Conversely, let $\{x_k\}_{k\in\mathbb{Z}}$ satisfy Equation (5.3) and set $y_k = -\partial_1 S(x_k, x_{k+1})$, for all integer k. Then, applying Equations (5.2) and (5.3) :

$$\begin{aligned} F(x_{k-1}, y_{k-1}) &= F \circ \psi^{-1}(x_{k-1}, x_k) = (x_k, \partial_2 S(x_{k-1}, x_k)) \\ &= (x_k, -\partial_1 S(x_k, x_{k+1})) = (x_k, y_k). \end{aligned}$$

□

Equations (5.3) can be interpreted as "discrete Euler-Lagrange" equations for some action function on the space of sequences. Indeed, let F be the lift of a twist map of the cylinder, and $S(x, X)$ its generating function. Given a sequence of points $\{x_N, \ldots, x_M\}$, we can associate its *action* defined by:

$$W(x_N, \ldots, x_M) = \sum_{k=N}^{M-1} S(x_k, x_{k+1})$$

Corollary 5.5 (Critical Action Principle) *A sequence $\{x_N, \ldots, x_M\}$ is the projection of an orbit segment of F on the x-axis if and only if it is a critical point of W restricted to the subspace of sequences $\{w_N, \ldots, w_M\}$ with fixed endpoints: $w_N = x_N, w_M = x_M$.*

Proof. Given a sequence $\{x_N, \ldots, x_M\}$, introduce the sequences $y_k = -\partial_1 S(x_k, x_{k+1})$ and $Y_k = \partial_2 S(x_k, x_{k-1})$. In particular, $F(x_k, y_k) = (x_{k+1}, Y_k)$. If $\tilde{W}$ is the restriction of W to the set of sequences with fixed endpoints x_N and x_M, a direct calculation yields:

$$d\tilde{W}(x_N, \dots, x_M) = \sum_{k=N+1}^{M-1} (Y_{k-1} - y_k) dx_k.$$

Hence $\{x_N, \dots, x_M\}$ is a critical point for W if and only if $Y_{k-1} = y_k$, which is a rephrasing of Equation (5.3) , $i.e.$ the sequence $\{(x_N, y_N), \dots, (x_M, y_M)\}$ is an orbit segment. □

Exercise 5.6 Adapt Lemma 5.4 to a situation where the map F is a composition of different twist maps $F = F_k \circ \dots \circ F_1$ with generating functions $S_1, \dots, S_k$. Note that you do not need to assume that the F_i are either all positive twist or all negative twist. If they are, one calls F a positive (*resp.* negative) *tilt map*.

C. Periodic Orbits

Let F be the lift of a twist map f of the annulus $\mathbf{A}$, or cylinder $\mathcal{C}$. Suppose that an orbit $\{x_k, y_k\}_{k\in\mathbb{Z}}$ of F satisfies, for some integers m and n:

$$x_{k+n} = x_k + m \tag{5.4}$$

that is, $F^n(x_k, y_k) = T^m(x_k, y_k)$. Then $f^n(proj(x_k, y_k)) = proj(x_k, y_k)$, and thus the orbit of (x_0, y_0) is the lift of a periodic orbit of f. We say that a sequence $\{x_k\}$ satisfying (5.4) is a (m, n) *sequence*. We denote by $X_{m,n}$ the space of m, n sequences. $X_{m,n}$ can be seen as the n-dimensional affine subspace of $\mathbb{R}^{n+1}$ of equation $x_{1+n} = x_1 + m$. Hence $X_{m,n}$ can be parameterized by the variables $(x_1, \dots, x_n)$.

An orbit whose x projection is an (m, n) sequence is called a (m, n) *orbit*, or an orbit of *type* (m, n). Hence, under n iterates of F, points in a (m, n) orbit get translated by the integer m in the x direction. Down in the annulus, this can be interpreted as the orbit wrapping m times around the annulus in n iterates. Conversely, it is not hard to see that any periodic orbit of f of period n lifts to an (m,n) orbit of a lift F, for some integer m which *does* depend on the choice of the lift F.

Proposition 5.7 *A (m, n) periodic sequence $\boldsymbol{x}$ is the x-projection of a m, n periodic orbit if and only if its is a critical point of*

$$W_{mn}(x_1, \dots, x_n) \stackrel{def}{=} S(x_n, x_1 + m) + \sum_{j=1}^{n-1} S(x_j, x_{j+1}).$$

We will sometimes refer to W_{mn} as the *periodic action*. The proof of Proposition 5.7 is very similar to that of Corollary 5.5.

Exercise 5.8 Show by an example that the number m for a periodic orbit of a twist map depends on the lift.

Exercise 5.9 Prove Proposition 5.7.

D. Rotation Numbers

Another interpretation of the numbers m, n in the lift of a periodic orbit is that the average displacement per iterate in the x direction of the points in a (m, n) orbit is m/n. In general, if $\{x_k, y_k\}_{k\in\mathbb{Z}}$ is *any* orbit, one can try to compute the limits:

$$\lim_{k\to+\infty} \frac{x_k}{k}, \qquad \lim_{k\to-\infty} \frac{x_k}{k}$$

If these limits exist, they are called respectively the *forward* and *backward rotation numbers*. If they are equal, they are called the *rotation number*. Since $\lim_{k\to\infty} \frac{x_k}{k} = lim_{k\to\infty} \frac{x_k - x_0}{k}$, the rotation number is an asymptotic measure of the average displacement per iterate in the x direction along an orbit. Obviously, an (m, n) periodic orbit has rotation number m/n. We also call *rotation number of the point* $z = (x, y)$ the rotation number of its orbit under F; we denote this number by $\rho_f(z)$.

Exercise 5.10 For those who know Birkhoff's ergodic theorem, show that, if f is an area preserving map of the annulus, $\rho_f(z)$ exists for a set of points z of full Lebesgue measure in $\mathcal{A}$ (*Hint.* $\lim \frac{x_k - x_0}{k} = \lim \frac{1}{k}\sum_1^k (x_j - x_{j-1})$ is the time average of some function. See Section 49).

6. Examples

A. The Standard Map

As noted in the introduction, one of the most widely studied family of monotone twist maps is the so called standard family, or *standard map*. We show how to retrieve explicitly the standard map from its generating function. Let

$$S(x, X) = \frac{1}{2}(X - x)^2 + V(x),$$

where V is 1-periodic in x. Define

$$y = -\partial_1 S(x, X) = X - x - V'(x)$$
$$Y = \partial_2 S(x, X) = X - x.$$

then it is easily seen that

$$X = x + Y$$
$$Y = y + V'(x),$$

That is, S generates the lift of a twist map:

$$F(x, y) = (X, Y) = (x + y + V'(x), y + V'(x)).$$

Taking as "potential" V the 1-parameter family $\frac{k}{4\pi^2} cos(2\pi x)$, we do indeed get the standard family:

$$F_k(x, y) = (x + y - \frac{k}{2\pi} sin(2\pi x), y - \frac{k}{2\pi} sin(2\pi x))$$

When $V \equiv 0$ (or k is equal to 0 in the standard family), the generating function is $\frac{1}{2}(X - x)^2 = \frac{1}{2}\mathrm{Dis}^2(x, X)$ and the map it generates is the *shear map*:

$$F_0(x, y) = (x + y, y)$$

which is *completely integrable*, in the sense that each horizontal line $\{y = y_0\}$ (covering a circle in $\mathcal{C}$) is invariant under F_0, and that the restriction of F_0 to $\{y = y_0\}$ is a translation: $x \mapsto x + y_0$ (lift of a rotation of angle $2\pi y_0$). We will see in Chapter 7 that F_0 is the time 1 map of the geodesic flow for the Euclidean metric on the circle.

As noted in the introduction, an important question about the standard family (or any set of maps containing a completely integrable one) is: which features of F_0 survive as one perturbs the parameter k away from 0?

Exercise 6.1 Check that the standard map satisfies all the axioms of twist maps of the cylinder.

B. Elliptic Fixed Points of Area Preserving Maps

Twist maps appear naturally in Hamiltonian systems when one consider the Poincaré return map around a periodic orbit. We will learn more about these return maps in Section 40. For now, it will suffice to say that, in an autonomous Hamiltonian system with two degrees of freedom, a small surface transverse to a periodic orbit gives rise to an area preserving map. For more on the use of these maps in Celestial mechanics, which was their original motivation, see Siegel & Moser (1971).

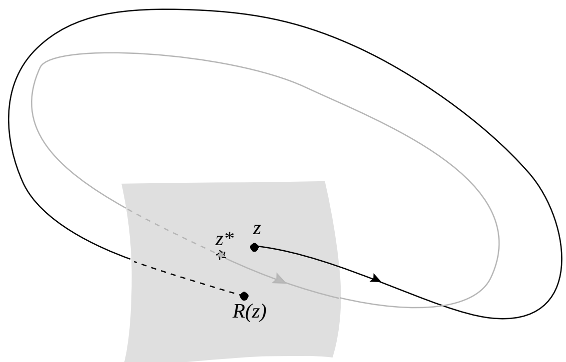

Fig. 6.1. A Poincaré section around the periodic orbit of the point z^*, with the return map R. The ambient 3-D space in this picture represents the energy level of a 2 degree of freedom Hamiltonian.

Let f be a symplectic C^∞ diffeomorphism in a neighborhood of 0 in $\mathbb{R}^2$, which has 0 as a fixed point. Since $\det\ Df(0) = 1$, the two eigenvalues are either real $\lambda, 1/\lambda$ or complex $\lambda, \overline{\lambda}$ and conjugated on the unit circle. In the first case, we say that 0 is a *hyperbolic fixed point*, in the second case that it is an *elliptic fixed point* (see Appendix 1). If f is the return map of a periodic orbit based at z^* as above, the periodic orbit is called elliptic or (resp. hyperbolic) when z^* is an elliptic (resp. hyperbolic) fixed point for R.

Suppose now that 0 is an elliptic fixed point and that $Df(0)$ has eigenvalues $\lambda = e^{i2\pi\alpha}$ and $\overline{\lambda}$ (i.e. Df(0) is a rotation of angle α). Suppose moreover that $\lambda^n \neq 1$ for n in $\{1, \dots, q\}$ for some integer q. We can make a change of variable $z = x + iy, \overline{z} = x - iy$ and write the Taylor expansion of order n of $f(z)$ in these coordinates:

$$f(z) = \sum_{k=1}^{n} R_k(z, \overline{z}) + o(|z|^n)$$

Theorem 6.2 (Birkhoff Normal Form) *There exists a symplectic (for the form $dx \wedge dy$), C^∞ diffeomorphism h, defined near 0 and having 0 as a fixed point such that:*

$$h \circ f \circ h^{-1}(z) = \lambda z e^{i2\pi P(z\overline{z})} + o(|z|^{q-1})$$

or, in polar coordinates ($z = re^{i2\pi\theta}$):

$$\tilde{f} = \overline{h} \circ f \circ \overline{h}^{-1}(r, \theta) = \big(\theta + \alpha + P(r^2) + o(|r|^{2n}),\ r + o(|r|^{2n})\big)$$

where $P(x) = a_1 x + \ldots + a_m x^m$ with $2m+1 < q$. Each of the "Birkhoff invariants" a_k is generically non zero.

For a proof of this, we refer to LeCalvez (1990) . *Generically* means that it is satisfied on a set of maps which is the intersection of countably many dense and open sets of C^∞ symplectic maps. There are also versions (see Moser (1973)) that require less differentiability. The point of this theorem is that, if we make the generic assumption that some a_k is non zero, the map f satisfies a twist condition in a neighborhood of $r = 0$ (for $r > 0$). Note that, in polar coordinates, the map $\tilde{f}$ preserves the form $rd\theta \wedge dr$, (which is only non–degenerate for $r > 0$. By making a further change of variables that preserves the vertical foliation $\{x = ct\}$, one can get a map that preserves $d\theta \wedge dr$ (see Chenciner (1985)). This last map preserves no boundaries. However, one can extend it to a boundary preserving map of a compact annulus. The main results in the theory can often be made precise enough to tell apart the dynamics of the original map from that of the extension. Hence *the dynamical study around conservative fixed points reduces to the study of twist maps.*

C. The Frenkel-Kontorova Model

The variational approach in Section 5 was encountered by Aubry (see Aubry & Le Daeron (1983)) while studying the Frenkel-Kontorova model in condensed matter physics. In this model, one considers a chain of particles whose nearest neighbor interaction is represented by springs (with spring constants 1). If x_k represents the location of the kth particle of the chain, the potential for the interaction between the k^{th} and $(k-1)^{\text{th}}$ particles is thus $\frac{1}{2}(x_k - x_{k-1})^2$. The chain of particles lies on the surface of a linear crystal represented by a periodic potential $V(x) = a/4\pi^2 cos(2\pi x)$ (a is some positive parameter).

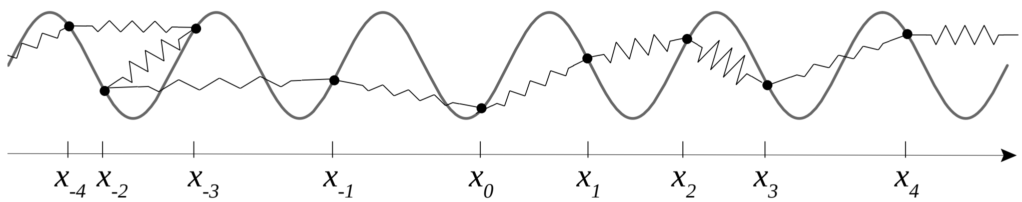

Fig. 6.3. The Frenkel–Kontorova Model.

The k^{th} particle is in equilibrium whenever the sum of the forces applied to it is null:

$$(x_{k+1} - x_k) - (x_k - x_{k-1}) - \frac{a}{2\pi} sin(2\pi x_k) = 0 \tag{6.1}$$

This equation can be rewritten $dW = 0$ where W, the energy of the configuration of particles is given by :

$$W = \sum_k S(x_k, x_{k+1}) = \sum_k \frac{1}{2}(x_k - x_{k+1})^2 + \frac{a}{4\pi^2} cos(2\pi x_k).$$

We recognize S as the generating function of the Standard Map. Hence *equilibrium states of the Frenkel-Kontorova model are in 1-1 correspondence with orbits of the Standard Map.*

D. Billiard Maps

We revisit here the example of the billiard map presented in the introduction. Consider the dynamics of a ball in a convex, planar billiard. This ball is subject to simple laws : it travels in straight lines between two rebounds and the incidence and reflexion angles are equal at a rebound. We reproduce here a figure of the introduction:

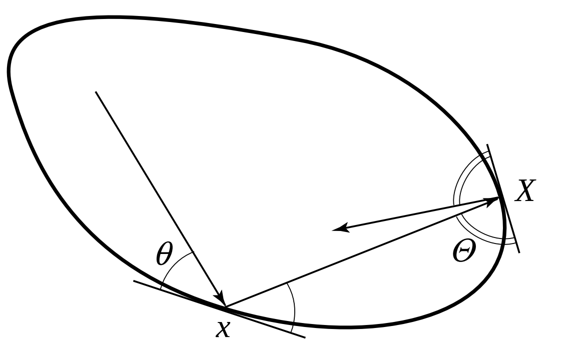

Fig. 6.4. In a convex billiard, the point x and angle θ at a rebound uniquely and continuously determines the next point X and incidence angle Θ.

Let x be the arc length coordinate with respect to a given point on the boundary C of the billiard, which we orient counterclockwise. Let $y = -cos(\theta)$ where θ is the reflexion angle of a point of rebound. Because of the convexity of the billiard and the law of reflexion, a pair (x, y) at a rebound determines its successor (X, Y) continuously, and vice versa. Hence we have constructed a homeomorphism $f : (x, y) \mapsto (X, Y)$ of the (open) annulus $\mathbb{S}^1 \times (-1, 1)$ which is actually a C^{k-1} diffeomorphism if the boundary is C^k (LeCalvez (1990)). We call f the *billiard map*. If we increase y while keeping x fixed, the convexity of C implies that $C(X)$ moves in the positive direction along C. Thus:

$$\frac{\partial X}{\partial y} > 0 \tag{6.2}$$

and the billiard map satisfies the positive twist condition.

We now show that f is exact symplectic by exhibiting a generating function for it. Let $S(x, X) = -\|C(X) - C(x)\|$ then, since $C' = \frac{dC}{dx}$ is a unit tangent vector:

$$\begin{aligned} \frac{\partial S}{\partial x} &= \frac{1}{S(x,X)}\left[C'(x).(C(x) - C(X))\right] = -y \\ \frac{\partial S}{\partial X} &= \frac{-1}{S(x,X)}\left[C'(X).(C(X) - C(x))\right] = Y \end{aligned} \tag{6.3}$$

which is to say:

$$YdX - ydx = dS(x, X) \tag{6.4}$$

Thus, for the billiard map, the action function $W = \sum S(x_k, x_{k+1})$ is nothing more than (minus) the length of the trajectory segment considered. For instance, periodic trajectories correspond to polygons in a given m, n-type who are critical points for the perimeter function. See the introduction for some illustrations.

Exercise 6.3 Show that the billiard map for the round billiard is given by $f(x, y) = (x + 2\cos^{-1}(-y), y)$.

Exercise 6.4 Show that, for the billiard map, the equation $dW = 0$ expresses the equality between the angle of incidence and the angle of reflexion at each rebound.

7. The Poincaré-Birkhoff Theorem

In this section, we give a complete proof of the Poincaré-Birkhoff theorem, also called Poincaré's last geometric theorem. We refer to Section 3 for some motivation for this theorem. We follow here the proof of LeCalvez (1991) . Even though this proof uses material scattered in various places in the book, in a first, light reading, it can serve as a good motivation to the methods used throughout the book. We use here some material on circle diffeomorphisms, which the reader can look up in the appendix to Chapter 2. We also use topological techniques of Conley for the estimation of number of critical points of the action function from Appendix 2. Finally, this proof was the inspiration behind Theorem 43.1.

We consider a map f of the compact annulus $\mathbf{A} = \mathbb{S}^1 \times [0, 1]$ and its lift F to $\mathcal{A} = \mathbb{R} \times [0, 1]$. We do *not* assume that f is a twist map, but rather that F satisfies the *boundary twist condition:* the restriction of $F\big|_{u_\pm}$ of F to each boundary component $u_- = \mathbb{S}^1 \times \{0\}$, $u_+ = \mathbb{S}^1 \times \{1\}$ which are lifts of circle diffeomorphisms, have rotation numbers $\rho_\pm$ which satisfy $\rho_- < \rho_+$ (See Section 13 for circle homeomorphisms and their rotation numbers).

Theorem 7.1 (Poincaré-Birkhoff) *The lift F of an area preserving map of $\mathcal{A}$ which satisfies the boundary twist condition. If $m/n \in [\rho_-, \rho_+]$, and m, n are coprime then F has at least two m, n–orbits. In particular, if $\rho_- < 0 < \rho_+$, then F has at least two fixed points.*

Proof. We first derive the general case from the special case. Suppose $F\big|_{u_\pm}$ has rotation numbers $\rho_\pm$ and that $m/n \in (\rho_-, \rho_+)$. Consider the map $G(\cdot) = F^n(\cdot) - (m, 0)$. Then G is area preserving and has new rotation numbers on the boundary $n(\rho_- - m/n) < 0 < n(\rho_+ - m/n)$. Hence G has at least two fixed points and they correspond to m, n periodic orbits of F. The proof of existence of fixed points of LeCalvez (1991) is based on the following lemma:

Lemma (Decomposition) 7.2 *Any area preserving map f of a bounded annulus $\mathbf{A}$ isotopic to the Identity, can be written as a composition of twist maps:*

$$f = f_{2K} \circ \ldots \circ f_1$$

Proof. It is a general fact (an open-closed argument, see Exercise 7.4) about topological groups that, given any neighborhood U of the neutral element of the group, any element in the connected component of the neutral element can be written as a finite products of elements of U. Let f_0 be the shear map $f_0(x, y) = (x + y \bmod 1, y)$ of the annulus. Since the set of maps satisfying the twist condition is open, there is a neighborhood U of Id in the set of area preserving maps of $\mathbf{A}$ which is such that $f \in U \Rightarrow f_0^{-1} \circ f$ is a negative twist map . Hence any f in U can be written as: $f = f_0 \circ (f_0^{-1} \circ f)$, a composition of two twist maps (one positive, the other negative). The group of area and orientation preserving maps of the annulus being connected (see Exercise 7.4), any map in that group can be written as a finite combinations of f as above. □

Let $F : \mathcal{A} \to \mathcal{A}$ be a lift of an area preserving map f of the compact annulus $\mathbf{A}$. Applying the previous lemma, we write $F = F_{2K} \circ \ldots \circ F_1$ where F_k lifts a twist map f_k. Let S_k be the generating function for F_k. We let

$$W_0(\boldsymbol{x}) = \sum_{k=1}^{2K} S_k(x_k, x_{k+1}), \qquad \boldsymbol{x} \in X_{0,2K} = \{\boldsymbol{x} \in \mathbb{R}^{2K+1} \mid x_{2K+1} = x_1\}.$$

Proposition 5.7 and its extension in Exercise 5.9 show that the critical points of W_0 correspond to periodic orbits under the successive f_k's, and hence to fixed points of f. To find these critical points, we study the gradient flow ζ^t of $-W_0$, solution of $\dot{\boldsymbol{x}} = -\nabla W_0(\boldsymbol{x})$. We exhibit a compact set P of $X_{0,2K}$ which must contain critical points for the action. The set P is an *isolating block* in the sense of Conley, *i.e.* a compact neighborhood whose boundary points immediately exit P in small positive or negative time (see Appendix 2). This condition on the boundary implies that the maximum invariant set for ζ^t is in the interior of P (hence the term "isolating").

Lemma 7.3 *Whenever $\rho_- < 0 < \rho_+$, the set*

$$P = \{\boldsymbol{x} \in X_{0,2K} \mid 0 \le -\partial_1 S_k(x_k, x_{k+1}) \le 1, \forall k \in \{1, \ldots, 2K\}\}$$

is an isolating block for the negative gradient flow ζ^t of $-W_0$. Moreover,

$$P \simeq \mathbb{S}^1 \times [0,1]^K \times [0,1]^{K-1}$$

with exit set $P^- = \mathbb{S}^1 \times [0,1]^K \times \partial([0,1]^{K-1})$

Proof. Setting $y_k = -\partial_1 S_k(x_k, x_{k+1})$, the faces of the boundary ∂P of P can be written as $\{y_k = 0\}$ or $\{y_k = 1\}$ for $k \in \{1, \ldots, 2K\}$. The behavior of the negative gradient flow at a face $y_k = 1$, say, is given by the sign of $\frac{dy_k}{dt} = \dot{y}_k$:

$$\dot{y}_k = -\frac{d}{dt}\left(\partial_1 S_k(x_k, x_{k+1})\right) = -\partial_{11} S_k(x_k, x_{k+1})\dot{x}_k - \partial_{12} S_k(x_k, x_{k+1})\dot{x}_{k+1}. \tag{7.1}$$

We let $Y_k = \partial_2 S_k(x_k, x_{k+1})$, *i.e.* $F_k(x_k, y_k) = (x_{k+1}, Y_k)$. With this notation $-\frac{\partial W_0}{\partial x_k} = Y_{k-1} + y_k$, and Equation (7.1) reads:

$$\dot{y}_k = \partial_{11} S_k(x_k, x_{k+1})(Y_{k-1} - y_k) + \partial_{12} S_k(x_k, x_{k+1})(Y_k - y_{k+1}) \tag{7.2}$$

and the invariance of the boundary component $\mathbb{R} \times \{1\}$ of $\mathcal{A}$ under F_k tells us that, since $y_k = 1$ then $Y_k = 1$ as well. Since $y_{k\pm 1} \leq 1$ and hence $Y_{k-1} \leq 1$,

$$Y_{k-1} - y_k \leq 0, \quad Y_k - y_{k+1} \geq 0. \tag{7.3}$$

Assume that k is even. Then f_k is a positive twist map and $\partial_{12} S_k(x_k, x_{k+1}) < 0$. We need to determine the sign of $\partial_{11} S(x_k, x_{k+1})$ on the subset $\{y_k = 1\}$ of ∂P. On this set, we have $x_k = a(x_{k+1})$ where a is the restriction of F_k^{-1} to $y = 1$, this latter set being parameterized by x. Since a is the lift of an orientation preserving circle diffeomorphism, we have $a'(x) > 0$ for all x. We differentiate the equation $1 = \partial S(a(x), x)$ with respect to x :

$$0 = a'(x)\partial_{11} S(a(x), x) + \partial_{12} S(a(x), x)$$

from which we deduce that $\partial_{11} S(x, a(x)) > 0$. Going back to Equation (7.2) , we see that if we are on the face $y_k = 1$ but away from its boundary (i.e., in particular, $y_l \neq 1,\ l = k-1, k+1$), then the inequalities in (7.3) are strict, and we get $\dot{y}_k < 0$: the flow is strictly entering P through this face, or exiting it in negative time.

If we are on an edge of the face $y_k = 1$, the inequalities (7.3) may be equalities. But this cannot be the case for all k: if it were, $(x_k)_{k\in\mathbb{Z}}$ would be critical and (x_k, y_k) would be a fixed point for f on the boundary, which is impossible since then the rotation number ρ_+ would be 0, a contradiction to $\rho_- < 0 < \rho_+$. So we can assume, say $Y_{l-1} - y_l < 0$, $y_l = y_{l+1} = \ldots = y_k = 1$, in which case (7.2) tells us that $\dot{y}_l \neq 0$ and the flow exits P in either positive or negative time at this point of ∂P.

The proof of the case k odd is exactly similar. We let the reader show in Exercise 7.5 that P and its exit set P^- have the topology advertised. □

This Lemma puts us in a situation which has become classic since the work of Conley & Zehnder (1983) in the field of symplectic topology. It can be schematized by the following diagram:

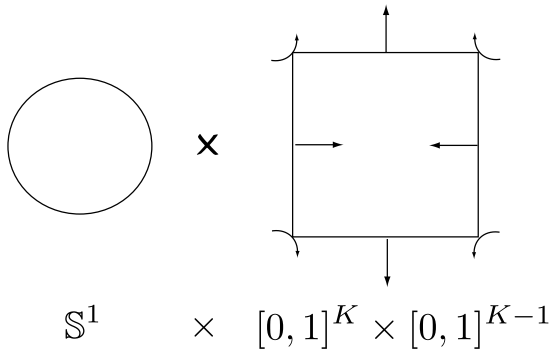

Fig. 7.2. The gradient flow at the boundary of the isolating block P

Given this topological behavior of the gradient flow, Proposition 62.4 tells us that W_0 must have at least $cl(\mathbb{S}^1) = 2$ critical points. This completes the proof of the Poincaré-Birkhoff Theorem. □

Exercise 7.4 Fill the arguments in the proof of Lemma 7.2 .
a) Show that the set of products of elements in any neighborhood of the neutral element in a topological group contains the connected component of the neutral element [*Hint.* Show that this set is open and closed].
b) Show that the set of continuous area and orientation preserving maps of the annulus $\mathbb{S}^1 \times [0,1]$ is connected [*Hint.* Show that the set of lifts of such maps is in fact convex. You might want to work with C^1 maps first, and argue by density for C^0 maps.]

Exercise 7.5 Show that the isolating block P is homeomorphic to $\mathbb{S}^1 \times [0,1]^K \times [0,1]^{K-1}$.