Appendix 2

SOME TOPOLOGICAL TOOLS

In order to estimate the minimum number of periodic orbits for a symplectic twist map or a Hamiltonian system, we need an estimate on the minimum number of critical points for the energy function of the corresponding variational problem. Estimating the number of critical points of functions on compact manifolds is the jurisdiction of Morse Theory and Lyusternick-Schnirelman Theory. Given the gradient flow of a real valued function f on a compact manifold M, Morse Theory rebuilds M from the unstable manifolds of the critical points of f. The combinatorial data of this construction gives a relationship between the set of critical points and the topology of M, in the guise of its homology. Unfortunately, the space on which the energy function W is defined is not compact. However, it usually is a vector bundle over a compact manifold M, and reasonably natural boundary conditions on the map or Hamiltonian system translates into some conditions of "asymptotic hyperbolicity" for W. This is a situation where Conley's theory, which studies the relationship between the recurrent dynamics of general flows and the topology of (pieces of) their phase spaces was brought to bear with great success.

For the reader who has no background in algebraic topology, we start in Section 60 by outlining, following some examples, an easy way to compute the homology of a manifold by decomposing it into cells. We then illustrate Morse theory in Section 61, by considering the cells given by the unstable manifolds of critical points of a real valued function on the manifold. We hope that this will give such a reader at least a flavor of the rest of this chapter. Starting in Section 62, we assume familiarity with algebraic topology. We give some of the basic definitions of Conley's theory and state results on estimates of number of critical points in isolated invariant sets for gradient flows. In Section 63, we prove most of these results. In Section 64, we apply these results to functions on vector bundles whose gradient flow are asymptotically hyperbolic.

60.* Hands On Introduction To Homology Theory

To a manifold, or to certain subspaces of it, we want to associate some algebraic objects called homology groups that are invariant under homeomorphisms or other natural topological deformations. Usually, the best way to calculate these groups (but not the best way to show their invariance properties), is to decompose the spaces studied into well understood pieces, and then define the groups from the combinatorial data describing how these pieces fit together. In this introduction we decompose spaces into *cells,* which are discs of different dimensions, and show how to compute cellular homology.

A. Finite Cell Complexes

Given a topological space X (e.g. a differentiable manifold) we can construct a new one by *attaching a cell* of dimension n. This is done by choosing an *attaching map* f from the bounding sphere $\mathbb{S}^{n-1}$ of the *cell* D^n (a disk of dimension n) to X. The new space, denoted by $X \cup_f D^n$ is given by the union of X and D^n where each point of ∂D^n is identified with its image by f in X. The topology on $X \cup_f D^n$ is that of the quotient $X \cup D^n / \{x \sim f(x)\}$.

Example 60.1 One can construct the sphere $\mathbb{S}^2$ by attaching the disc D^2 to a point p. The space $X = \{p\}$ is a manifold of dimension 0, and the attaching map f sends each point of the boundary circle of D^2 to p. One can also construct a sphere by attaching a disk to another one (what is the attaching map?). These constructions have obvious generalization to higher dimensions.

A *cellular space* is a space built by successively attaching a finite number of cells, starting from a finite number of points (cells of dimension 0). If in this process each cell is attached to cells of lower dimensions, the space obtained is called a *finite cell complex* or *CW complex.* The union of all cells of dimension less than k in a finite cell complex is called the k–*skeleton.* Thus the $k+1$–skeleton is built by attaching cells of dimension $k+1$

to the k–skeleton. The dimension of the cell of maximum dimension in a cellular space X (and hence of a CW complex) is called the *dimension* of X.

Examples 60.2 Figure 60.1 shows how the torus can be decomposed (this is not the only way!) into a finite cell complex: its 0–skeleton is the point $\boldsymbol{z}$. To get the 1–skeleton we attach both extremities of the "equator" $\boldsymbol{a}$ and the "meridian" $\boldsymbol{b}$ to $\boldsymbol{z}$. The attaching maps send the boundaries -1 and 1 of the 1–cells $\boldsymbol{a} \cong [-1,1] \cong \boldsymbol{b}$ to the point $\boldsymbol{z}$. Finally, the 2–skeleton is obtained by attaching the disk D (stretched to a square) to the 1–skeleton as indicated by the "flat" picture of the torus. Note that the 1–skeleton looks like a "bouquet of two circles".

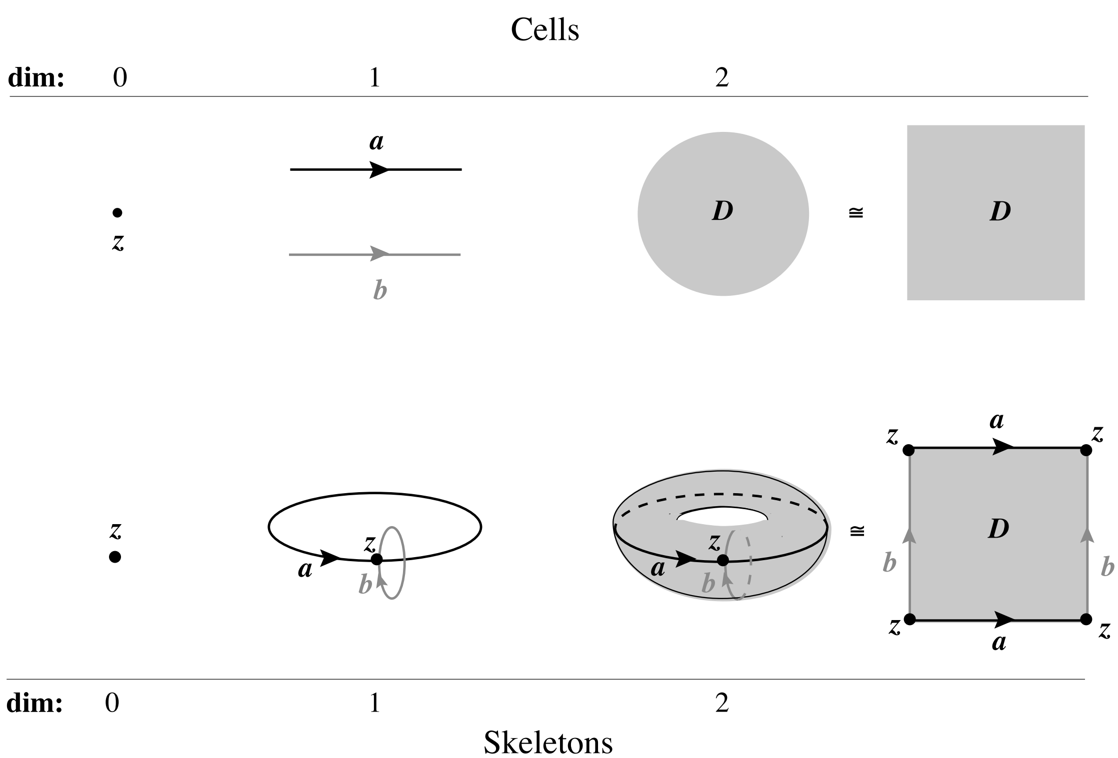

Fig. 60.1. The torus $\mathbb{T}^2$ as a finite CW complex.

One can generalize this construction to surfaces of any genus g (spheres with g handles) by gluing a 2 cell to a polygon with $4g$ sides and identifying all vertices to a single point, and edges two by two as indicated by their name and orientations on the following figure (g is 2 here):

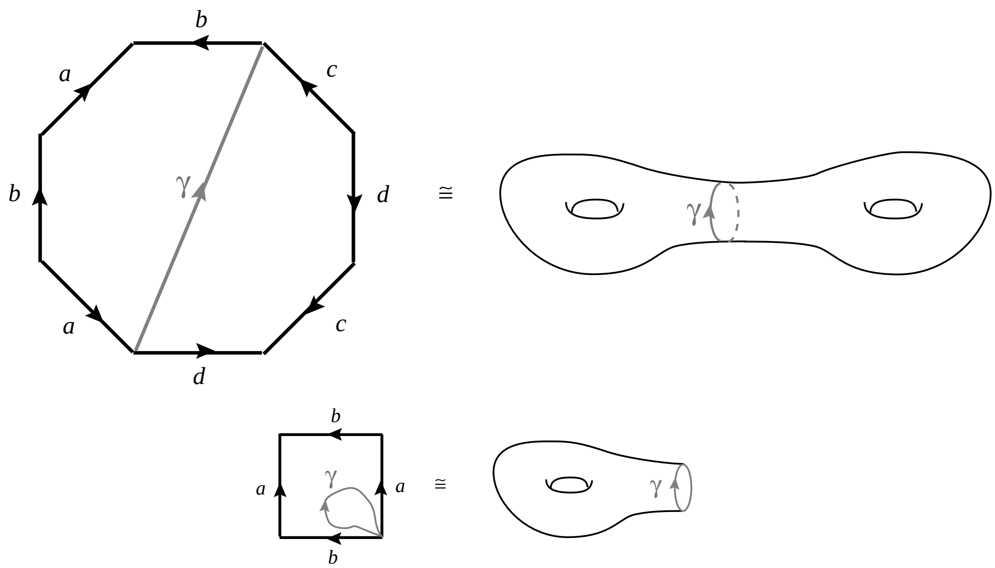

Fig. 60.2. The double torus (surface of genus 2) as a finite CW complex. Identify edges according to their names and orientations, and identify all vertices to one point. When cutting the octagon in half through the curve γ we obtain two *handles*, which are tori with a disk (bounded by the curve γ) removed in each. Any compact surface can be described with this method.

More generally, we will show in the next section that any compact manifold is homeomorphic to a finite cell complex.

Exercise 60.3 Decompose $\mathbb{S}^n, \mathbb{T}^n, \mathbb{RP}^n$ and the Klein bottle into finite cell complexes. Remember that $\mathbb{RP}^n$ can be defined as $\mathbb{D}^n / \sim$, where the relation $\sim$ identifies any two antipodal points on the boundary of the n–disk $\mathbb{D}^n$. The *Klein bottle* is $[-1,1]^2/\sim$ where $(1,y) \sim (-1,-y)$ and $(x,1) \sim (x,-1)$.

B*. Cellular Homology

Bouquets of spheres. When we "crush" the (k-2)–skeleton X_{k-2} of a finite cell complex X to a point inside the (k-1)–skeleton X_{k-1}, the boundary of each (k-1)-cell crushes to that point. Hence each (k-1)-cell of X_{k-1} becomes a (k-1)-sphere in X_{k-1}/X_{k-2}. All these spheres meet at exactly one point, where the crushed X_{k-2} collapsed: we say that X_{k-1}/X_{k-2} is a *bouquet of spheres*. The attaching map f of a k–cell to X_{k-1} gives rise to a map $\tilde{f} : \mathbb{S}^{k-1} \to X_{k-1}/X_{k-2}$, by composition with the quotient map. Hence we have a map $\tilde{f}$ from a sphere of dimension $k-1$ to a bouquet of spheres, all of dimension $k-1$.

Digression on degree and homotopy. Any continuous map from a sphere S_1 to a sphere S_2 of same dimension comes equipped with a *degree* which, informally, is an integer which measures the number of times S_1 "wraps around" S_2 under this map. This integer can be negative, as we keep track of orientation. Since the proper topological definition of degree requires homology (which we are in the process of defining), we restrict ourselves to differentiable maps. The degree of a differentiable map f between two manifolds of same dimension is given by:

$$deg(f) = \sum_{\boldsymbol{x} \in f^{-1}(\boldsymbol{z})} 1 \cdot (sign \det Df_{\boldsymbol{x}}) \tag{60.1}$$

where $\boldsymbol{z}$ is any *regular* value of f, $i.e.$ no determinant in the above sum is zero (by Sard's theorem, almost all values of a smooth map are regular). It turns out that the above number is independent of the (regular) point $\boldsymbol{z}$. The degree of a map is also invariant under homotopy of the map. [Two continuous maps f_0 and f_1 between the manifold M and the manifold N are *homotopic* if there is a continuous map (called *homotopy*) $F : [0,1] \times M \to N$ such that $F(0, \boldsymbol{z}) = f_0(\boldsymbol{z}), \quad F(1, \boldsymbol{z}) = f_1(\boldsymbol{z})$ for all $\boldsymbol{z}$ in M].

Back to horticulture. The attaching map $\tilde{f} : \mathbb{S}^{k-1} \to X_{k-1}/X_{k-2}$ has a multiple degree: on each sphere S_i in the bouquet one can compute the oriented number of preimages under $\tilde{f}$ of a regular point as in (60.1) (without loss of generality, we can assume that $\tilde{f}$ is differentiable except at the common point of the spheres). Suppose that $\boldsymbol{c}_1^{k-1}, \ldots, \boldsymbol{c}_{N_{k-1}}^{k-1}$ denote the $(k-1)$–cells of the cell complex and $\boldsymbol{c}_1^k, \ldots, \boldsymbol{c}_{N_k}^k$ its k-cells. We now form an $N_k \times N_{k-1}$ integer matrix ∂_k whose entry $\partial_k(ij)$ is the degree of the attaching map from $\partial \boldsymbol{c}_i^k$ to the jth sphere of the bouquet, i.e. $\boldsymbol{c}_j^{k-1}/\partial \boldsymbol{c}_j^{k-1}$. The matrices ∂_k, for $k \in \{1, \ldots, dimX\}$ essentially give all the combinatorial data describing how the complex X is pieced together from our collection of cells.

Chain complexes. We now want to view the matrices ∂_k as those of linear maps between finite dimensional vector spaces, or modules. To do this, one thinks of $\boldsymbol{c}_1^k, \ldots, \boldsymbol{c}_{N_k}^k$ as the basis vectors of an abstract vector space (or free module) C_k whose elements are formal sums of the form

$$\boldsymbol{c} = \sum_1^{N_k} a_j \boldsymbol{c}_j^k ,$$

where a_j is an element of some "coefficient" field (or ring) K (usually $\mathbb{Z}_2, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$). Hence C_k is generated by the k-cells and $dim C_k = N_k$. For convenience, we define $\partial_0 \equiv 0$ on C_0.

Lemma 60.4

$$\partial_{k-1} \circ \partial_k \equiv 0. \tag{60.2}$$

The proof of this crucial lemma, which we will not give here (see, *eg.* Dubrovin & al. (1987)) usually uses the long exact sequence of a triple and a pair in simplicial homology. A chain of maps and vector spaces (or modules):

$$C_n \overset{\partial_n}{\to} C_{n-1} \to \dots C_k \overset{\partial_k}{\to} C_{k-1} \to \dots \to C_0$$

satisfying (60.2) is called a *chain complex.*

Definition 60.5 The kth *homology group* of the finite cell complex X with coefficients in a ring or field K is given by:

$$H_k(X; K) = Ker\ \partial_k / Im\ \partial_{k+1}.$$

where, by convention, $\partial_0 = 0 = \partial_{n+1}$

This definition makes sense since, by Lemma 60.4, $Im\ \partial_{k+1} \subset Ker\ \partial_k$. Note that $H_k(X) = 0$ whenever $k > \dim X$ or $k < 0$, since for such k, $X_k = \emptyset$.

Homology Of The Circle. The circle $\mathbb{S}^1$ is a CW complex: we start with a point p and attach to it an interval $I = [0, 1]$; the boundary points of I become identified to p under the attaching map. Using $\mathbb{R}$ as coefficients in our chain complex, we get $C_0 = \mathbb{R}.p \cong \mathbb{R}, C_1 = \mathbb{R}.I \cong \mathbb{R}$. The map $\partial_1 \equiv 0$: p has the two preimages $\{0\}$ and $\{1\}$ under the attaching map, but they come with opposite orientations under the orientation induced by I. Hence the degree of the attaching map is 0 here and the homology of the circle is given by:

$$H_k(\mathbb{S}^1, K) = \begin{cases} K \text{ if } k = 0, 1, \\ 0 \text{ otherwise} \end{cases}.$$

Homology Of The Torus. Figure 60.1 above gives the generators for a chain complex for the torus: $C_0 = \mathbb{R}\cdot \boldsymbol{z}, C_1 = \mathbb{R}\cdot \boldsymbol{a} \oplus \mathbb{R}\cdot \boldsymbol{b}, C_2 = \mathbb{R}\cdot D$. All the boundary maps are 0 in this case: $\partial_1 \boldsymbol{a} = 0$ because it geometrically yields $\boldsymbol{z}$ twice but with opposite orientation. Likewise for $\partial_1 \boldsymbol{b}$. As for $\partial_2 D = \boldsymbol{a} + \boldsymbol{b} - \boldsymbol{a} - \boldsymbol{b} = 0$, again due to orientation. Hence $Ker\ \partial_k / Im\ \partial_{k+1} = C_k$ for $k = 1, 2, 3$. We have shown:

$$H_k(\mathbb{T}^2, \mathbb{R}) \cong \begin{cases} \mathbb{R} & k = 0, 2 \\ \mathbb{R}^2 & k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, this result remains valid if we replace $\mathbb{R}$ by any coefficient ring K.

Homology Of The K lein BottleA less trivial example is given by the Klein bottle. This non orientable surface is a torus with a twist and it cannot be embedded in $\mathbb{R}^3$. We build it with the same cells $\boldsymbol{z}, \boldsymbol{a}, \boldsymbol{b}$ and D as the torus. The only change occurs in the definition of ∂_2: instead of gluing D to two copies of $\boldsymbol{b}$ in opposite orientation, we give them the same orientation (according to that of the boundary of D):

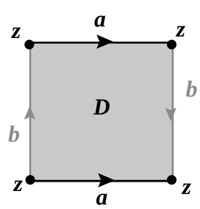

Fig. 60.3. A cell decomposition for the Klein bottle. The only difference with that of the torus is the orientation of one of the segments $\boldsymbol{b}$.

Let us use the integers $\mathbb{Z}$ as our coefficient ring. As a result of the orientation change, the matrix of ∂_2 is now $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $Ker\ \partial_2 = \{0\}$. From this we immediately get that $H_2(Klein, \mathbb{Z}) = 0$. As in the case of the torus, $\partial_1 \equiv 0$ so that $Ker\ \partial_1 = C_1 = \boldsymbol{a}\cdot\mathbb{Z} \oplus \boldsymbol{b}\cdot\mathbb{Z}$. Since $Im\ \partial_2 = \{0\}\cdot \boldsymbol{a} \oplus 2\mathbb{Z}\cdot \boldsymbol{b},\ H_1(Klein, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. As in the case of the torus, $H_0(Klein, \mathbb{Z}) = \mathbb{Z}$ (in fact, the rank of H_0 gives the number of connected components of a manifold). Now let's reexamine the above computation with coefficients $K = \mathbb{Z}_2$ instead: the map $\partial_2 = 0$ in this case since 2=0 in this ring. Thus, in this case we are back to the same situation as with the torus: $H_0(Klein, \mathbb{Z}_2) \cong \mathbb{Z}_2, H_1(Klein, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, H_2(Klein, \mathbb{Z}_2) \cong \mathbb{Z}_2$. Finally, let's choose $K = \mathbb{R}$. Since $Ker\ \partial_2 = C_2$ in this case again, $H_2(Klein, \mathbb{R}) = \mathbb{R}$. Since $\mathbb{R}/2\mathbb{R} = \mathbb{R}/\mathbb{R} = \{0\}, H_1(Klein, \mathbb{R}) \cong \mathbb{R}$. As before $H_0(Klein, \mathbb{R}) \cong \mathbb{R}$.

Some general properties and definitions related to homology. Let X be a compact manifold of dimension n. As we will see in next section, it can always be decomposed into a finite CW complex.

•$dim H_k(M, \mathbb{R}) = rank\ H_k(M, \mathbb{Z}) = b_k$ is the k^{th} *Betti number* of M.

•$\sum_{k=1}^{n} (-)^k b_k = \chi(M)$ is the *Euler characteristic* of M.

•Neither b_k nor $\chi(M)$ depend on the chain decomposition chosen for M.

•b_0 gives the number of connected component of M.

•$b_n = 1$ if M is orientable, $b_n = 0$ if M is not orientable.

•The importance of homology stems in great part from its *invariance under topological equivalences*. One topological equivalence is that of homeomorphism. A coarser equivalence (see Exercise 60.7) is that of homotopy type. Two topological spaces M and N have the same *homotopy type* if there are continuous maps $\phi : M \to N$, $\psi : N \to M$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are homotopic to the Identity maps of M and N respectively. In other words M can be deformed into N and vice-versa.

Theorem 60.6 *If the two manifolds M and N are homeomorphic, or have the same homotopy type, then they have same homology: $H_*(M) = H_*(N)$ (the star $*$ stands for any integer).*

C. Cohomology

Roughly speaking, cohomology is dual to homology. For readers of this book, it might be easier to see it through differential forms, which are dual to chains of cells in the sense that the integral $< c, \omega >= \int_c \omega$ of a form ω on a chain c is a linear, real valued function of the variable c (it is also linear in ω). The duality bracket given by integration also satisfies:

$$< \partial\, c, \omega >=< c, d\omega >$$

where d is the exterior differentiation on forms. This formal equality is a general requirement for defining cohomology. In the case of forms it is simply given by Stokes' Theorem. Finally, we can define the cochain complex

$$0 \to C_0^* \xrightarrow{d_1} C_1^* \xrightarrow{d_2} \ldots \xrightarrow{d_n} C_n^* \to 0$$

where $C_k^* = \Lambda^k$ is the vector space of k–forms and d_k is exterior differentiation. As with homology, we can define the *DeRham cohomology group* as:

$$H^k(M, \mathbb{R}) = Ker\ d_{k+1}/Im\ d_k,$$

$i.e.$ this cohomology is the quotient of closed forms over exact forms. One notable difference between homology and cohomology is the direction of the arrows in the complexes that defines them. Another notable difference, which makes the use of cohomology often preferable, is the existence of a natural product operation in cohomology, called the *cup product.* In DeRham cohomology, this cup product takes the form of wedge product of the forms:

$$[\omega_1] \cup [\omega_2] = [\omega_1 \wedge \omega_2]$$

where the notation $[\omega]$ denotes the class of the closed form ω and $\cup$ the cup product. There are many different ways to define cohomology, but it can be shown that (given some normalization requirements), they all give the same result on compact manifolds. Poincaré, for instance, introduced cohomology (not under that name) by geometrically constructing a dual complex to a triangulation (a special CW chain decomposition). In the next section, where unstable manifolds of critical points of a Morse function provide us with a chain decomposition, the dual decomposition can be taken to be that of stable manifolds.

Exercise 60.7 Show that the circle and the cylinder have same homotopy type but are not homeomorphic.

Exercise 60.8 Using Exercise 60.3, compute the homology of $\mathbb{S}^n, \mathbb{T}^n, \mathbb{RP}^n$.

61.* Morse Theory

We now show how any compact manifold can be described as a cellular space, with cells given by the unstable manifolds of the critical points of a Morse function. This immediately yields a relationship between critical points and homology, in the guise of the Morse Inequalities. We first define some of these terms. For more details on the material in this section, the reader should consult Milnor (1969) .

Gradient Flow And Morse Functions. Let $f : M \to \mathbb{R}$ be a differentiable function on a manifold M. A *critical point* for f is a point $\boldsymbol{z}$ at which the differential of f is zero: $df(\boldsymbol{z}) = 0$. If f is twice differentiable, the critical point $\boldsymbol{z}$ is called *nondegenerate* if

$$\det \frac{\partial^2 f(\boldsymbol{z})}{\partial \boldsymbol{x}^2} \neq 0 \tag{61.1}$$

where this second derivative is taken with respect to any local coordinates $\boldsymbol{x}$ around $\boldsymbol{z}$ on M. The function f is a *Morse function* if all its critical points are nondegenerate. One can show that, on any given manifold, Morse functions are generic in the set of twice differentiable functions (see *eg.* Guillemin & Pollack (1974) or Milnor (1969)). Note that Condition (61.1) is independent of the coordinate system. Indeed, at a critical point $\boldsymbol{z}$,

$$\frac{\partial^2 f(\boldsymbol{z})}{\partial \boldsymbol{y}^2} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{y}}^t \frac{\partial^2 f(\boldsymbol{z})}{\partial \boldsymbol{x}^2} \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}.$$

This last formula also implies that the number of negative eigenvalues of the real, symmetric matrix $\frac{\partial^2 f(\boldsymbol{z})}{\partial \boldsymbol{x}^2}$ does not depend on the coordinate system chosen around the critical point $\boldsymbol{z}$. This number is called the *Morse index* of $\boldsymbol{z}$. Qualitatively, the level set portrait of a function around a nondegenerate critical point is entirely determined by the index of the critical point. Indeed:

Lemma 61.1 (Morse Lemma) *Let $\boldsymbol{z}$ be a nondegenerate critical point for a function f on a manifold of dimension n. There is a coordinate system $\boldsymbol{x}$ around $\boldsymbol{z}$ such that:*

$$f(\boldsymbol{x}) = f(\boldsymbol{z}) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2.$$

We refer the reader to Milnor (1969) for a proof of this lemma, which generalizes the diagonalization process (Gram-Schmidt) for bilinear forms. Since the Morse Lemma clearly implies that the critical points of a Morse functions are isolated, we have:

Corollary 61.2 *A Morse function on a compact manifold has a finite number of critical points.*

The *gradient flow* of a function f is the solution flow for the O.D.E.:

$$\dot{\boldsymbol{z}} = -\nabla f(\boldsymbol{z}). \tag{61.2}$$

The *gradient* ∇f is defined here by $\langle \nabla f, . \rangle = df(.)$, where the brackets denotes some chosen Riemannian metric (on $\mathbb{R}^n$ or $\mathbb{T}^n$, one usually uses the dot product). f decreases along the flow:

$$\frac{d}{dt} f(\boldsymbol{z}(t)) = -\left\{\nabla f(\boldsymbol{z}(t))\right\}^2 \leq 0$$

with equality occurring exactly at the critical points. The eigenvectors corresponding to the negative eigenvalues of $\frac{\partial^2 f(z)}{\partial x^2}$ span a subspace of $T_{\boldsymbol{z}} M$ which is tangent to the unstable manifold at $\boldsymbol{z}$ of the gradient flow. [We remind the reader that the *unstable manifold* of a restpoint for a flow ϕ^t is the manifold $W^u(\boldsymbol{z}_*)$ of points whose backward orbit is asymptotic to the restpoint $\boldsymbol{z}_*$: $W^u(\boldsymbol{z}_*) = \{\boldsymbol{z} \mid \lim_{t \to -\infty} \phi^t(\boldsymbol{z}) = \boldsymbol{z}_*\}$]. In the case of the gradient flow of a Morse function, the unstable manifold of a critical point is tangent to the $x_1, \ldots, x_k$ plane given by the Morse Lemma. Hence *the Morse index of a nondegenerate critical point of a Morse function is the dimension of its unstable manifold*.

Remark 61.3 Note that if the metric chosen to define the gradient is the euclidean one in the Morse coordinate chart, the $(x_1, \ldots, x_k)$ plane is itself the (local) unstable manifold of the critical point, at least in that chart. This can always be arranged, by a local perturbation of the metric, and we will assume from now on that this is the case.

Topology Of Sublevel Sets. The gist of Morse theory consists in studying how the topology of the *sublevel set*:

$$M^a = \{\boldsymbol{x} \in M \mid f(\boldsymbol{x}) \leq a\}$$

changes as a varies.

Theorem 61.4 *If there are no critical points in $f^{-1}[a,b]$, then M^a and M^b are diffeomorphic. The inclusion of M^a in M^b is a deformation retraction.*

Proof. Deform M^b into M^a by flowing down the trajectories of the gradient flow, with appropriate speed (controlled by reparameterization of the flow) and during an appropriate time interval. This is possible as long as there are no critical value in $[a,b]$. □

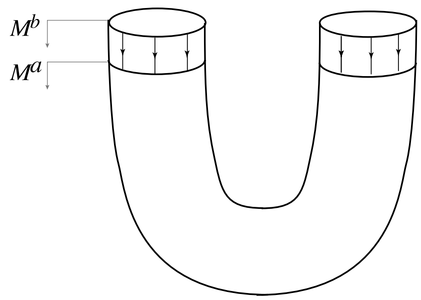

Fig. 61.1. Deformation of a sublevel set M^b into the sublevel set M^a when there are no critical points in $f^{-1}[a,b]$. The lines with arrows represent trajectories of the gradient flow.

Theorem 61 .5 *Suppose $f^{-1}[a,b]$ is compact and has exactly one critical point in its interior, which is degenerate and of index k. Then M^b has the homotopy type of M^a with a cell of dimension k attached, namely, a ball in the unstable manifold of the critical point.*

Proof. (sketch) Let $\boldsymbol{z}$ be the critical point, $c = f(\boldsymbol{z})$ and $\epsilon > 0$ be a small real number. By the previous theorem, $M^{c+\epsilon}$ has the same homotopy type as M^b and likewise for $M^{c-\epsilon}$ and M^a. Hence, we just have to show that $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a cell attached.

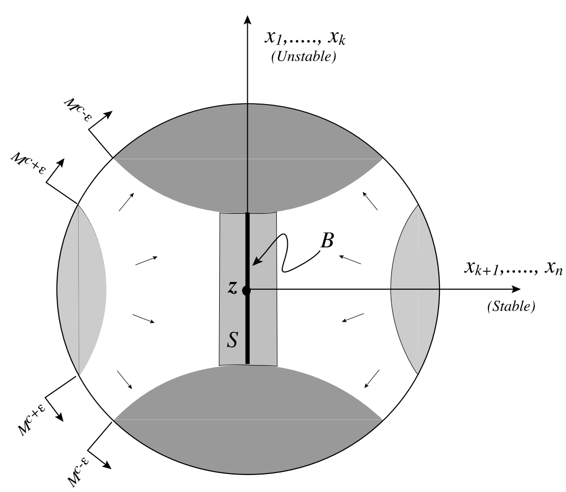

Fig. 61.1. A neighborhood of a Morse critical point $\boldsymbol{z}$. A suitable parameterization of the flow retracts $M^{c+\epsilon}$ onto $M^{c-\epsilon} \cup S$, which itself can be deformed into $M^{c-\epsilon} \cup B$.

We have represented in Figure 61.1 the sets $M^{c\pm\epsilon}$ within a Morse neighborhood. The drawing makes it intuitively clear that some reparameterization of the gradient flow (which we have represented by some arrows) will collapse $M^{c+\epsilon}$ into $M^{c-\epsilon}\cup S$, where the set S is given by:

$$S=\{f\leq c+\epsilon, x_{k+1}^2+\ldots+x_{k+n}^2\leq\delta\}.$$

S can obviously be deformed into:

$$B=\{f\leq c+\epsilon, x_{k+1}=\ldots=x_{k+n}=0\},$$

that is, a ball in the unstable manifold of $\boldsymbol{z}$. In other words,

$$M^{c+\epsilon}\simeq M^{c-\epsilon}\cup B.$$

□

Any cellular space X is homotopically equivalent to a finite cell (CW) complex Y, where X and Y have the same number of cells in each dimension (one deforms each of the attaching maps defining X into one that attaches to cells of lower dimensions than its own, see Dubrovin & al. (1987), Section 4). This and the previous theorems yield:

Theorem 61.6 *Any sublevel set M^a of a Morse function on a compact manifold M has the homotopy type of a finite CW complex, whose cells correspond to the unstable manifolds of the critical points.*

Morse Inequalities. Since there always is a Morse function on any given manifold, Theorem 61.6 yields:

Corollary 61.7 *Any compact manifold has the homotopy type of a finite CW complex.*

Theorem 61.8 (Morse Inequalities) *Given any Morse function f on a compact manifold M, the homology of M is generated by a finite complex $\{C_k,\partial_k\}_{\{1,\ldots,dimM\}}$ whose generators correspond to the critical points of index k of f. In particular, if $c_k=dimC_k$ is the number of critical points of index k,*

$$c_k\geq b_k=\dim H_k(M,\mathbb{R}) \tag{61.3}$$

and, better:

$$c_k - c_{k-1} + \ldots \pm c_0 \geq b_k - b_{k-1} + \ldots \pm b_0, \tag{61.4}$$

with equality holding for $k = n$.

Proof. The first statement in the theorem is somewhat of a tautology for us, since we have defined the homology of M as the cellular homology of any cellular complex representing M, and thus in particular, we can choose the complex generated by the unstable manifolds of the critical points. Formula (61.3) is then trivial, since

$$H_k(M) = Ker\ \partial_k / Im\ \partial_{k+1},$$

and $Ker\ \partial_k$ is a subspace of C_k. The inequalities (61.4) are a consequence of (61.3) and their proof, left to the reader, only involves linear algebra. □

Floer-Witten Complex. One can give a nice geometric interpretation to the maps ∂_k in the context of Morse theory. Assume that the gradient flow ϕ^t of our chosen Morse function is *Morse-Smale*, i.e. that for any given pair of critical points $\boldsymbol{x}, \boldsymbol{z}$, their respective stable and unstable manifold meet transversally. This is again a generic situation, which has the following implications: the set

$$M(\boldsymbol{x}, \boldsymbol{z}) = W^u(\boldsymbol{x}) \cap W^s(\boldsymbol{z}),$$

which is the union of all orbits connecting $\boldsymbol{x}$ and $\boldsymbol{z}$, is a manifold and

$$dim M(\boldsymbol{x}, \boldsymbol{z}) = index(\boldsymbol{x}) - index(\boldsymbol{z}).$$

In particular, if $index(\boldsymbol{x}) - index(\boldsymbol{z}) = 1$, $M(\boldsymbol{x}, \boldsymbol{z})$ is a one dimensional manifold made of a finite number of arcs that one can count, with $\pm$ according to a certain rule of intersection. This intersection number $m(\boldsymbol{x}, \boldsymbol{z})$ gives the coefficient in the generator $\boldsymbol{z}$ of $\partial(\boldsymbol{x})$, i.e.

$$\partial_k \boldsymbol{x} = \sum_{\boldsymbol{z} \in C_{k-1}} m(\boldsymbol{x}, \boldsymbol{z}).\boldsymbol{z}.$$

One can also define cohomology in this fashion: just take the same complex, but defined for the function $-f$. What was a stable manifold becomes an unstable manifold and C_k

becomes C_{n-k}. This not only gives us a geometric way to see cohomology, but a trivial proof of Poincaré's duality theorem for compact manifolds:

$$H^{n-k}(M,\mathbb{R}) \cong H_k(M,\mathbb{R}).$$

For more details on this chain complex, which is sometimes called the Floer-Witten complex but dates back to J. Milnor's book on cobordism, see *eg.* Salamon (1990). For a proof of Poincaré's duality using the Morse complex, see Dubrovin & al. (1987).

62. Controlling the Topology of Invariant Sets

The relationship revealed by Morse between the critical point data of a function and the topology of the underlying manifold has a very wide generalization in the theory of Conley, which brings about a similar relationship for general continuous flows on locally compact topological spaces. The relationship is between local topological data of recurrent parts of the flow and the global topology of well chosen regions of the space. We will outline this theory in Section 63.C. For now, we make a small step toward this generalization.

Here, and for the rest of this chapter, the cohomology used is the Čech cohomology with coefficients in $\mathbb{R}$ (see Spanier (1966)). We do not need to define this cohomology here: it is enough to state that it is well defined on compact sets. Furthermore it has a continuity property: If $X = \cap X_n$, then $H^*(X) = \lim_{\to} H^*(X_n)$, where the latter expresses the inductive limit of the cohomology modules. Otherwise, Čech cohomology satisfies all the usual axioms and rules of cohomology and coincides with other cohomologies on compact manifolds. In this section, we present some of the fundamental concepts and results of Conley's theory, which are used in this book. The next section goes into more detail and provides proofs for the theorems presented in this section, as well as more results.

Isolating Blocks. Consider a compact set I which is invariant under the gradient flow of a function W on some finite dimensional manifold. If W is a Morse function, then necessarily I is made of critical points and the intersections of all their stable and unstable manifolds (prove it as an exercise!). Exactly as we did for manifolds, consider the Floer-Witten chain complex, generated by the critical points and with boundary maps given by the stable-unstable manifolds intersection data. It turns out (see Floer (1989a), or Salamon (1990)) that this complex gives the cohomology Conley Index of I, a topological/dynamical invariant of I that we define below. In certain cases, as in what follows, one can evaluate the cohomology Conley index and hence give lower estimates on the number of critical points. We use these results in Section 64 to estimate the number of critical points of functions on vector bundles.

Definition 62.1 Let M be a finite dimensional manifold. A compact neighborhood B in M is called an *isolating block* for a (continuous) flow ϕ^t if points on the boundary ∂B of B immediately leave B under the flow, in positive or negative time:

$$\boldsymbol{z} \in \partial B \Rightarrow \phi^{(0,\epsilon)} \subset B^c \quad \text{or} \quad \phi^{(-\epsilon,0)} \subset B^c \quad \text{for some} \quad \epsilon = \epsilon(\boldsymbol{z}) > 0.$$

The *exit set* B^- of B is defined as the set of points in ∂B which immediately flow out of B in $positive$ time. Given an isolating block B for the flow ϕ^t, define $I(B)$ to be the *maximal invariant set* included in B (“maximal” is in the sense of inclusion here). Alternatively, $I(B) = \cap_{t\in\mathbb{R}}\phi^t(B)$.

Cohomology Conley Index and Cuplength. There are two classical ways to measure the topological complexity of an invariant set $I(B)$. One is its *cohomology Conley index*:

$$h(I) = H^*(B, B^-).$$

The bigger the dimension of this vector space, the more complex the topology of I. Note that in the notation “$h(I)$”, we have deliberately omitted the mention of B: this is because the vector spaces $H^*(B, B^-)$ are isomorphic for all isolating block B such that $I = I(B)$ (Conley & Zehnder (1984)). Hence $h(I)$ is an invariant of the set I. In practice, the size of $h(I)$ is often measured by the *sum of the Betti numbers*

$$sb(h(I)) = \sum_k dim H^k(B, B^-).$$

This again is an invariant of I. A second, somewhat rougher way to measure the complexity of an invariant set I (or any topological space which admits continuous (semi)flows and a cohomology) is the *cuplength* which is defined as:

$$cl(I) = 1 + \sup\{k \in \mathbb{N} \mid \exists\, \omega_1, \ldots \omega_k,\ \omega_j \in H^{n_j}(I), n_j > 1,\ \text{and}\ \omega_1 \cup \ldots \cup \omega_k \neq 0\}$$

Generalized Morse and Lyusternick-Schnirelman Theories. The following results are consequences of the much more general theory of Conley for (semi)flows that we sketch in the next section.

Theorem 62.2 *Let I be a compact isolated invariant set for the gradient flow of a function W on some manifold. If the function is Morse, the number of critical points in I is greater or equal to $sb(h(I))$. In general, the number of critical points in I is at least equal to $cl(I)$.*

Remark 62.3 If one applies Theorem 62.2 to the case $I = M = B$, $B^- = \emptyset$, then $h(I) = H^*(M)$, $cl(I) = cl(M)$ and we retrieve both the critical point estimates of Morse and Lyusternick-Schnirelman.

Historically, the first time Theorem 62.2 was applied in a significant way was in the proof of the following proposition, which appeared in several parts in Conley & Zehnder (1983) :

Proposition 62.4 *Let M be a compact manifold and W be a real valued function on $M \times \mathbb{R}^n \times \mathbb{R}^m$. Suppose that the gradient flow of W admits an isolating block B of the form $B \simeq M \times D^+ \times D^-$ with exit set $M \times D^+ \times \partial D^-$, where $D^+ \subset \mathbb{R}^n, D^- \subset \mathbb{R}^m$ are homeomorphic to the unit balls. If W is a Morse function, it has at least $sb(M)$ critical points in B. In general, W has at least $cl(M)$ critical points.*

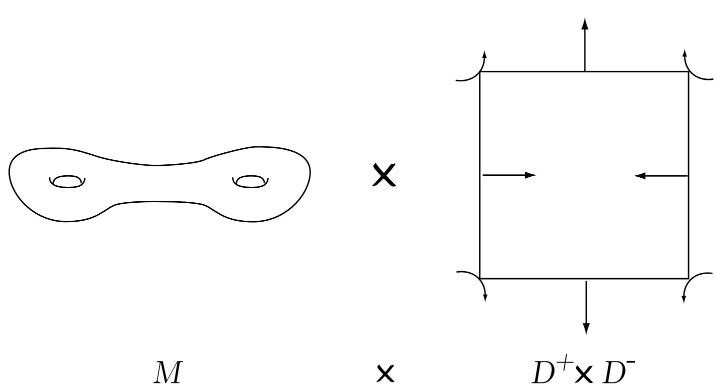

Fig. 62.0. The isolating neighborhood in Proposition 62.4.

Conley and Zehnder applied this theorem in the case $M = \mathbb{T}^n$, where $sb(\mathbb{T}^n) = 2^n$, and $cl(\mathbb{T}^n) = n+1$. In the following section we will give a proof of Theorem 62.2 (we will only sketch the sum of betti number estimate, but give a complete proof of the cuplength estimate) as well as of Proposition 62.4.

63. Topological Proofs

A. Proof of the Cuplength Estimate in Theorem 62.2

Conley & Zehnder (1983) prove a cuplength estimate (their Theorem 5) that is valid for a compact invariant set I of a general flow ϕ^t. We follow their proof. Define a *Morse decomposition* for I to be a finite collection $\{M_p\}_{p\in P}$ of disjoint compact and invariant subsets of I, which can be ordered in such a way that any x not in $\cup_{p\in P} M_p$ is α-asymptotic to an M_j and ω-asymptotic to an M_i, with $i < j$ (x is *α-asymptotic* (resp. *ω-asymptotic*) to M_j if $\lim_{t\to-\infty\ (resp.\ +\infty)} \phi^t(x) \in M_j$). One can show that a compact invariant set always has such a Morse decomposition. We now state Theorem 5 of Conley & Zehnder (1983) :

Theorem 63.1 (Conley-Zehnder) *Let I be any compact invariant set for a continuous flow, and let $\{M_P\}_{p\in P}$ be a Morse decomposition for I. Then*

$$cl(I) \leq \sum_{p\in P} cl(M_p). \tag{63.1}$$

The relevant example for us is when ϕ^t is the gradient flow of a function with a finite number of (not necessarily nondegenerate) critical points on a compact invariant set I: it easy to check that these critical points form a Morse decomposition. Since an isolated point has trivial cohomology, $cl(M_p) = 1$ for each p in this example, and *we have proven the cuplength estimate in Theorem 62.2*. The case when the critical points are not isolated is trivial in that theorem: $cl(I) < \infty$ is always true...

Proof of Theorem 63.1. Note that if $(M_1, \ldots, M_k)$ is a Morse decomposition, then $(M_{1,\ldots,k-1}, M_k)$ is also a Morse decomposition, where $M_{1,\ldots,k-1}$ is formed by the union of $M_1 \cup \ldots \cup M_{k-1}$ and of all the connecting orbits between these sets. Hence, by induction, we only need to consider the case where $k = 2$, and (M_1, M_2) is a Morse decomposition for I. From the definition of a Morse decomposition, we can deduce the existence of two compact neighborhoods I_1 of M_1 and I_2 of M_2 in I with $I_1 \cup I_2 = I$ and such that $M_1 = \cap_{t>0}\phi^t(I_1)$ and $M_2 = \cap_{t<0}\phi^t(I_2)$. In particular, by continuity of the Čech cohomology $H^*(I_j) = H^*(M_j), j = 1, 2$. Thus the proof of (63.1) reduces to that of the inequality $cl(I_1) + cl(I_2) \geq cl(I)$ whenever $I_1 \cup I_2 = I$ are three compact sets. The next lemma is devoid of dynamics:

Lemma 63.2 *Let $I_1 \cup I_2 \subset I$ be three compact sets. If $i_1 : I_1 \to I$, $i_2 : I_2 \to I$ and $i : I_1 \cup I_2 \to I$ are the inclusion maps, then, for any $\alpha, \beta \in H^*(I)$,*

$$i_1^*\alpha = 0 \quad \text{and} \quad i_2^*\beta = 0 \Rightarrow i^*(\alpha \cup \beta) = 0.$$

Proof. We chase the diagram:

$$\begin{array}{ccccc} H^*(I, I_1) & \otimes & H^*(I, I_2) & \stackrel{\cup}{\to} & H^*(I, I_1 \cup I_2) \\ \downarrow j_1^* & & \downarrow j_2^* & & \downarrow j^* \\ H^*(I) & \otimes & H^*(I) & \stackrel{\cup}{\to} & H^*(I) \\ \downarrow i_1^* & & \downarrow i_2^* & & \downarrow i^* \\ H^*(I_1) & \otimes & H^*(I_2) & \stackrel{\cup}{\to} & H^*(I_1 \cup I_2). \end{array}$$

The vertical sequences are exact sequences of pairs, the horizontal lines are given by Künneth Formula. Starting on the second line of the diagram with $\alpha, \beta \in H^*(I)$, suppose $i_1^*\alpha = 0 = i_2^*\beta$ then there must be $\tilde\alpha \in H^*(I, I_1)$ with $j_1^*\tilde\alpha = \alpha$, $\tilde\beta \in H^*(I, I_2)$ with $j_2^*\tilde\beta = \beta$. Now $j^*(\tilde\alpha \cup \tilde\beta) = \alpha \cup \beta$ and hence $i^*(\alpha \cup \beta) = i^* \circ j^*(\tilde\alpha \cup \tilde\beta) = 0$, by exactness. □

To finish the proof of Theorem 63.1, let $\alpha_1, \ldots, \alpha_l$ be in $H^*(I)$ and $\alpha_1 \cup \ldots \cup \alpha_l \neq 0$. Let this product be maximum, so that $cl(I) = l + 1$. Order the α's in such a way that $\alpha_1 \cup \ldots \cup \alpha_r$ is the longest product not in the kernel of i_1^*. In particular $cl(I_1) \geq r + 1$ and $i_1^*(\alpha_1 \cup \ldots \cup \alpha_r \cup \alpha_{r+1}) = 0$. Lemma 63.2 forces $i_2^*(\alpha_{r+1} \cup \ldots \cup \alpha_l) \neq 0$ (i^* is one-to-one here, since $I_1 \cup I_2 = I$). Thus $cl(I_2) \geq l - (r + 1) + 1 = l - r$, and $cl(I_1) + cl(I_2) \geq l + 1 = cl(I)$. □

B*. The Betti Number Estimate of Theorem 62.2 and Conley's Theory

We have proven in Theorem 63.1 that, for a general function W, the number of critical points in an invariant set I for the gradient flow of W is greater than $cl(I)$. We now show that if W is a Morse function, the number of critical points in I is greater than $sb(h(I))$. To do so, one can either follow Floer (1989a) in his generalization of the Witten complex (of unstable manifolds of critical points for gradient flows, see the end of Section 61) to invariant sets. His proof relies in part on Conley's theory. Alternatively, one can use Conley's generalized Morse inequalities that we state in this subsection.

Let I be a compact invariant set for a continuous flow ϕ^t on some locally compact topological space. Let $(M_1, \ldots, M_k)$ be a Morse decomposition for I. Analogously to the cuplength estimates of Theorem 63.1, Conley-Morse inequalities relate certain betti numbers of the Morse sets M_j to the corresponding betti numbers of I. To define the adequate betti numbers, we need to generalize the notion of isolating block to that of index pair for isolated invariant sets. A compact set I is an *isolated invariant set* if there is a neighborhood N of I such that $I = I(N)$ is the maximal invariant subset in N. An *index pair* for an isolated invariant set is a generalization of an isolating block. Roughly, I is a pair of compact spaces (N_1, N_2) such that $N_1 \backslash N_2$ is a neighborhood of I and $I = I(N_1 \backslash N_2)$. Moreover N_2 plays the role of the exit set: to exit $N_1 \cup N_2$, a point of N_1 must first go through N_2, see Conley (1978), Conley & Zehnder (1984). The fundamental property of these sets is that the homotopy type $[N_1/N_2, *]$ is independent of the choice of index pair for I and hence defines a topological invariant called the *Conley index* of the invariant set I. Giving less information, but easier to manipulate is the *cohomology Conley index* $H^*(N_1, N_2) = h(I)$, again an invariant of I. If $(N_1, N_2) = (B, B^-)$ for an isolating block B, this definition of $h(I)$ is the same as we have given in Section 62. One way to encode the information given by $h(I)$ is via the coefficients of the *Poincaré polynomial*:

$$p(t, h(I)) := \sum_{j \geq 0} t^j dim H^j(N_1, N_2).$$

In Conley & Zehnder (1984), it is proven that, given a Morse decomposition $(M_1, \ldots, M_k)$ for an invariant set I of a continuous flow ϕ^t, there is a *filtration* $N_0 \subset N_1 \subset \ldots \subset N_k$ such that (N_j, N_{j-1}) is an index pair for M_j. This is instrumental in proving the following (Conley & Zehnder (1984):

Theorem 63.3 (Conley-Morse Inequalities)

$$\sum_{j=1}^{k} p(t, h(M_j)) = p(t, h(I)) + (1+t)Q(t), \tag{63.2}$$

where $Q(t)$ is a polynomial with positive coefficients

This theorem is an extraordinary generalization of the classical Morse inequalities: it is valid for any *continuous* flow on a locally compact space (not necessarily a manifold!).

End of Proof of Theorem 62.2. To retrieve the betti number estimates of Theorem 62.2, we use the Morse decomposition of the invariant set I made of the (isolated) critical points $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_N$. Thanks to the Morse Lemma, we can construct an isolating block for each $\boldsymbol{z}_j$, and show that the Conley index of $\boldsymbol{z}_j$ has the homotopy type of a pointed sphere made by collapsing the boundary of the local unstable manifold of $\boldsymbol{z}_j$ to a point. The isolating block is denoted by S in Figure 63.1.

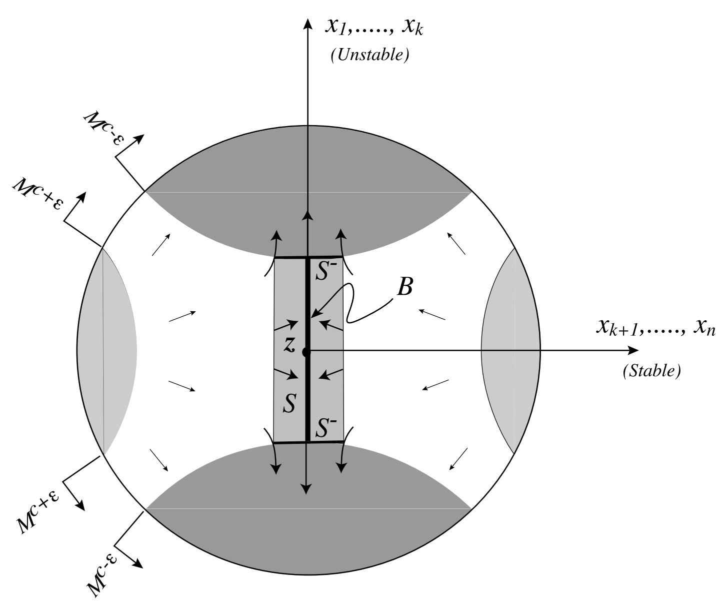

Fig. 63.1. The index pair (S, S^-) retracts on (B, B^-), a pair made of the local unstable manifold of $\boldsymbol{z}$ and its boundary (a disk of dimension k equal to the index of the critical point $\boldsymbol{z}$ and its bounding sphere). Thus $h(\boldsymbol{z}) = H^*(S, S^-) \cong H^*(B, B^-) \cong H^*(\mathbb{S}^k, *)$ which has exactly one generator in dimension k.

Hence $p(t, h(\boldsymbol{z}_j)) = t^{u_j}$, where u_j is the Morse index of $\boldsymbol{z}_j$. Now the pair $(I, \emptyset)$ is an isolating pair for I (no points exit I), and thus $p(t, h(I)) = \sum t^k dim H^k(I)$. The positivity of the coefficients of Q in (63.2) therefore insures that there are at least $dim H^k(I)$ critical points of index k in I. □

C. Floer Lemma

The following lemma, proven in Floer (1987) gives a situation where one can get a handle on the topology of an invariant set I. It is central to the proofs of several topological results we will use, including Proposition 62.4.

Lemma 63.4 (Floer Lemma) *Let B be an isolating block for a flow ϕ^t on a finite dimensional manifold, and I be its maximal invariant set. Suppose that there is a retraction $\alpha : B \to P$, where P is some compact subset of B. If there is a class $u \in H^*(B, B^-)$ such that :*

$$v \mapsto u \cup \alpha^*(v) : H^*(P) \to H^*(B, B^-)$$

is an isomorphism, then

$$\alpha_I{}^* : H^*(P) \to H^*(I)$$

is injective, where α_I denotes the restriction of α to I.

(If $N \subset M$ are two topological spaces and $i : N \to M$ is the inclusion map, a *retraction* is a map $r : M \to N$ such that $r \circ i = Id_N$, that is r restricts to Id on N).

Corollary 63.5 *Let B, I, P be as in Lemma 63.4, and let the flow ϕ^t in that lemma be the gradient of some function W. Then the number of critical points of W is at least $cl(P)$.*

Proof. Since $H^*(P) \to H^*(I)$ is injective, $cl(I) \geq cl(P)$ and the Corollary is an immediate consequence of Theorem 62.2. □

Proof of Lemma 63.4. Define $B^\infty = \cap_{t>0} \phi^t B$, the set of points that stay in B for all negative time.

Lemma 63.6 *1)* $H^*(B, B^\infty \cup B^-) = 0$
2) $l^* : H^*(B^\infty) \to H^*(I(B))$ *is an isomorphism, where* $l : I(B) \to B^\infty$ *is the inclusion.*

Before proving this lemma, we use it to finish the proof of Lemma 63.4. Consider the diagram:

$$\begin{array}{ccccccc}
H^*(B,B^-) & \otimes & H^*(B,B^\infty) & \overset{\cup}{\to} & H^*(B,B^\infty\cup B^-) & = 0 \\
\downarrow Id & & \downarrow j^* & & \downarrow k^* & \\
H^*(B,B^-) & \otimes & H^*(B) & \overset{\cup}{\to} & H^*(B,B^-) & \\
 & & \downarrow i^* & & & \\
 & & H^*(B^\infty) & \overset{l^*}{\overset{\sim}{\to}} & H^*(I) &
\end{array}$$

where all vertical maps are induced by inclusions, and the two first horizontal maps are given by Künneth Formula. Suppose ${\alpha_I}^* v = 0$ for some $v \in H^*(P)$. Since l^* is an isomorphism and $\alpha_I = \alpha_{B^\infty} \circ l$, $0 = {\alpha_I}^* = l^*(\alpha_{B^\infty})^* v \Rightarrow (\alpha_{B^\infty})^* v = 0$. Since $\alpha_{B^\infty} = \alpha \circ i$, $0 = {\alpha_{B^\infty}}^* v = i^* \alpha^* v$. The middle, vertical sequence is the exact sequence of a pair. Hence there is a $w \in H^*(B, B^\infty)$ such that $j^* w = \alpha^* v$. But $u \cup \alpha^* v = k^*(u \cup w) = k^*(0) = 0$. The hypothesis of Lemma 63.4 forces $v = 0$. □

Proof of Lemma 63.6. Let $B^t = \phi^t(B)$ and $B^\infty = \cap_{t>0} B^t$ as before. Note in particular that, in the Hausdorff topology, $\lim_{t\to+\infty} B^t = B^\infty$, and $\lim_{t\to 0} B^t = B$. To the triple of spaces $(B, B^t \cup B^-, B^-)$ corresponds the exact sequence:

(63.3)

$$\ldots \overset{\delta^*}{\to} H^*(B, B^t \cup B^-) \to H^*(B, B^-) \overset{i^*}{\to} H^*(B^t \cup B^-, B^-) \overset{\delta^*}{\to} H^{*-1}(B, B^t \cup B^-) \ldots,$$

(see *eg.* Dubrovin & al. (1987)). We now show that i^* is an isomorphism. Consider the diagram:

$$\begin{array}{ccc}
 & & (B^t \cup B^-, B^-) \\
 & \overset{i_1}{\nearrow} & \downarrow i \\
(B^t, B^- \cap B^t) & \overset{i_2}{\longrightarrow} & (B, B^-)
\end{array}$$

.

The excision theorem implies that i_1^* is an isomorphism, and the continuity of the Čech cohomology implies that i_2^* is an isomorphism. Since the diagram commutes, i^* must be an isomorphism. But this forces $H^*(B, B^t \cup B^-) = 0$ in (63.3). Taking the limit of this equality as $t \to \infty$ proves 2).

Using the long exact sequence of the pair (B^∞, I), the map l^* induced by the inclusion $l : I \to B^\infty$ is an isomorphism whenever $H^*(B^\infty, I) = 0$, which we proceed to show. Note that $\phi^{-t}B^\infty \subset B^\infty$ and, by definition, $I = \cap_{t\geq 0}\phi^{-t}(B^\infty)$. Consider the maps:

$$(B^\infty, \phi^{-t}B^\infty) \xrightarrow{\phi^{-t}} (\phi^{-t}B^\infty, \phi^{-t}B^\infty) \xrightarrow{j} (B^\infty, \phi^{-t}B^\infty),$$

where j is the inclusion. The map $j \circ \phi^{-t}$ is clearly homotopic to Id, hence

$$H^*(B^\infty, \phi^{-t}B^\infty) \cong H^*(\phi^{-t}B^\infty, \phi^{-t}B^\infty) = 0.$$

Since this is true for all t, the continuity of the Čech cohomology concludes. □

D. Proof of Proposition 62.4

To prove this proposition, we let the manifold M play the role of P in Lemma 63.4. The retraction α of that lemma is given here by the canonical projection $\alpha : B \to M$. Clearly the projection of B onto $M \times D^-$ is a deformation retract, which deforms B^- onto $M \times \partial D^-$. Hence $H^*(B, B^-) \cong H^*(M \times D^-, M \times \partial D^-)$. Now, Künneth Formula gives an isomorphism:

$$H^*(M) \otimes H^*(D^-, \partial D^-) \overset{\cup}{\xrightarrow{\sim}} H^*(M \times D^-, M \times \partial D^-)$$

where, as suggested by the notation, one gets all of the classes in the right hand side vector space as cup products of classes in the two left hand side spaces (with the appropriate identifications given by the inclusion maps). But, letting $n = dimD^-$, we have $H^*(D^-, \partial D^-) \cong H^*(\mathbb{S}^n, \cdot)$, which has exactly one generator u in dimension n.

Hence $H^{*+n}(B, B^-) \cong H^*(M)$ and $sb(h(I)) = sb(M)$ where I is the maximal invariant set in B. This and Theorem 62.2 yield the Betti number estimate. The homeomorphism $H^*(M) \cong H^*(B, B^-)$ is of the type prescribed by Lemma 63.4. This implies that the induced map $H^*(M) \to H^*(I)$ is injective and hence $cl(I) \geq cl(M)$. This fact and Theorem 62.2 give the cuplength estimate. □

E*. Floer's Theorem of Global Continuation of Hyperbolic Invariant Sets.

Floer's Lemma 63.4 is the cornerstone to the proof of the following theorem, where he makes good use of the powerful property of "invariance under continuation" of the Conley Index. This theorem illustrates the power of Conley's theory, and shows the historical root of Floer Cohomology. Note that, in the theory of dynamical systems, the hyperbolicity of an invariant set for a dynamical system is intimately related to its persistence under *small* perturbations of the system: this relationship is the core of many theorems on structural stability. What is interesting about the following theorem (and Conley's theory in general) is that it provides situations when the (rough) persistence of an invariant set can be proven *for arbitrarily big perturbations.*

The notion of continuation of invariant sets makes use of the simple following fact: an index pair for a flow ϕ^t will remain an index pair for all flows that are C^0 close to ϕ^t. Two isolated invariant sets for two different flows ϕ_0^t, ϕ_1^t are *related by continuation* if there is a continuous curve ϕ_λ^t of flows joining them which can be (finitely) covered by intervals (in λ) of flows having the same index pair. The following theorem appeared as Theorem 2 in Floer (1987) to which we refer the reader for a proof. It can be seen as an instance of weak, but *global*, stability of normally hyperbolic invariant sets, which we now define. An invariant set G is *normally hyperbolic* for a flow ϕ^t on a manifold N if there is a decomposition:

$$TN\big|_G = TG \oplus E^+ \oplus E^-$$

which is invariant under the covariant linearization of the vector field V corresponding to ϕ^t with respect to some metric $\langle\ ,\ \rangle$, so that for some constant $m > 0$:

$$\langle \xi, DV\xi\rangle \leq -m\langle \xi,\xi\rangle \ (resp. \geq +m\langle\xi,\xi\rangle) \text{ for } \xi \in E^- \ (resp.\ E^+) \tag{63.4}$$

Theorem 63.7 (Floer) *Let ϕ_λ^t be a continuous one parameter family of flows on a C^2 manifold N. Suppose that G_0 is a compact C^2 submanifold invariant under the flow ϕ_0^t. Assume that G_0 is normally hyperbolic for the flow ϕ_0^t and suppose that there is a retraction $\alpha : N \to G_0$. Finally, suppose that there is a family G_λ of invariant sets for ϕ_λ^t which are related by continuation to G_0. Then the map:*

$$\alpha\big|_{G_\lambda}^* : H^*(G_0) \to H^*(G_\lambda)$$

in Čech cohomology is injective.