Appendix 2 or TOPO

SOME TOPOLOGICAL TOOLS

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In order to estimate the minimum number of periodic orbits a symplectic twist map or a Hamiltonian system may have, we need an estimate on the minimum number of critical points for the energy function of the corresponding variational problem. Estimating the number of critical points of functions on compact manifolds is the jurisdiction of Morse Theory and Lyusternick-Schnirelman Theory. Given the gradient flow of a real valued function f on a compact manifold M, Morse Theory rebuilds M from the unstable manifolds of the critical points of f. The combinatorial data of this construction gives a relationship between the set of critical points and the topology of M, in the guise of its homology. Unfortunately, the space on which the energy function W is defined is not compact. However, it usually is a vector bundle over a compact manifold M, and reasonably natural boundary conditions on the map or Hamiltonian system translates into some conditions of "asymptotic hyperbolicity" for W. This is a situation where Conley's theory, which studies the relationship between the recurrent dynamics of general flows and the topology of (pieces of) their phase spaces was brought to bear with great success.

For the reader who has no background in Algebraic Topology, we start in Section 61 by outlining an easy way to compute the homology of a manifold by decomposing it into cells. We then illustrate Morse theory by considering the cells given by the unstable manifolds of critical points of a real valued function on the manifold. We hope that this will give such a reader at least a flavor of the rest of this chapter. Starting Section 63 , we assume familiarity with algebraic topology. We give the basic definitions of Conley's theory and state results on estimates of number of critical points in isolated invariant sets for gradient flows. In Section 64, we prove these results. In Section 65, we apply these results to functions on vector bundles whose gradient flow are asymptotically hyperbolic.

61.* Hands On Introduction To Homology Theory

To a manifold, or to certain subspaces of it, we want to associate some algebraic objects called homology groups that are invariant under homeomorphisms or other natural topological deformations. Usually, the best way to calculate these groups (but not the best way to show their invariance properties), is to decompose the spaces studied into well understood pieces, and then define the groups from the combinatorial data describing how these pieces fit together. In this introduction we decompose spaces into *cells*, which are discs of different dimensions, and show how to compute cellular homology.

A. Finite Cell Complexes

Given a topological space X (e.g. a differentiable manifold) we can construct a new one by *attaching a cell* of dimension n. This is done by choosing an *attaching map* f from the bounding sphere $\mathbb{S}^{n-1}$ of the *cell* D^n (a disk of dimension n) to X. The new space, denoted by $X \cup_f D^n$ is given by the union of X and D^n where each point of ∂D^n is identified with its image by f in X. The topology on $X \cup_f D^n$ is that of the quotient $X \cup D^n / \{x \sim f(x)\}$.

Examples 61.1 One can construct the sphere $\mathbb{S}^2$ by attaching the disc D^2 to a point p. The space $X = \{p\}$ is a manifold of dimension 0, and the attaching map f sends each point of the boundary circle of D^2 to p. One can also construct a sphere by attaching a disk to another one (what is the attaching map?). These constructions have obvious generalization to higher dimensions.

A *cellular space* is a space built by attaching a finite number of cells (successively), starting from a finite number of points (cells of dimension 0). If in this process each cell is attached to cells of lower dimensions, the space obtained is called a *finite cell complex* or *CW complex*. The union of all cells of dimension less than k in a finite cell complex is called the *k–skeleton*. Thus the $k+1$–skeleton is built by attaching cells of dimension $k+1$ to the k–skeleton. The dimension of the cell of maximum dimension in a cellular space X (and hence of a CW complex) is called the *dimension* of X, denoted by the usual $dim X$.

Examples 61.2 The torus can be decomposed (not in a unique way!) into a finite cell complex: its 0–skeleton is the point z. To get the 1–skeleton we attach both extremities of the "meridian" a and the "equator" b to z. The attaching maps send the boundaries -1 and 1 of the 1–cells $a \cong [-1, 1] \cong b$ to the point z. Finally, the 2–skeleton is obtained by attaching the disk D (stretched to a square) to the 1–skeleton as indicated by the "flat" picture of the torus. Note that the 1–skeleton looks like a "bouquet of two circles".

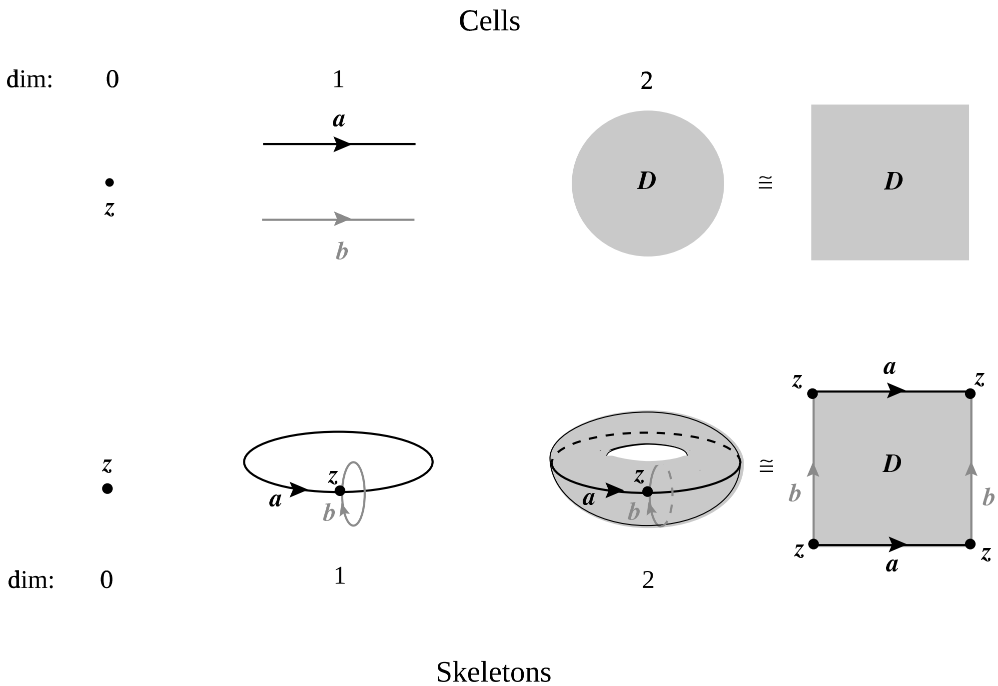

Fig. 61. 2. The torus $\mathbb{T}^2$ as a finite CW complex.

One can generalize this construction to surfaces of any genus g (spheres with g handles) by gluing a 2 cell to a polygon with $4g$ sides and identifying all vertices to a single point, and edges two by two as indicated by their name and orientations on the following figure (g is 2 here):

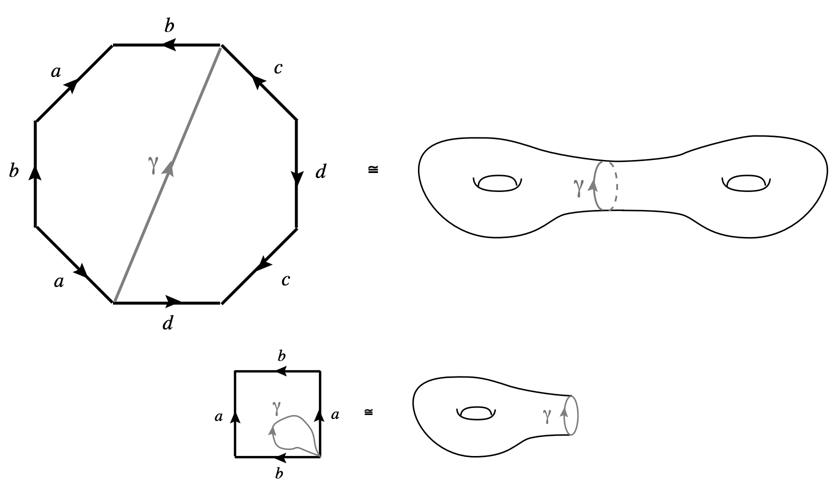

Fig. 61. 3. The double torus (surface of genus 2) as a finite CW complex. Identify edges according to their names and orientations, and identify all vertices to one point. When cutting the octagon in half through the curve γ we obtain two *handles*, which are tori with a disk (bounded by the curve γ) removed in each.

More generally, we will show in the next section that any compact manifold is homeomorphic to a finite cell complex.

Exercise 61.3 Decompose $\mathbb{S}^n, \mathbb{T}^n, \mathbb{RP}^n$ and the Klein bottle into finite cell complexes. Remember that $\mathbb{RP}^n$ can be defined as $\mathbb{D}^n/\sim$, where the relation $\sim$ identifies any two antipodal points on the boundary of the n–disk $\mathbb{D}^n$. The *Klein bottle* is $[-1,1]^2/\sim$ where $(1,y) \sim (-1,-y)$ and $(x,1) \sim (x,-1)$.

B. Cellular Homology

Bouquets of spheres. When we "crush" the (k-2)–skeleton X_{k-2} of a finite cell complex X to a point inside the (k-1)–skeleton X_{k-1}, the boundary of each (k-1)-cell crushes to that point. Hence each (k-1)-cell of X_{k-1} becomes a (k-1)-sphere in X_{k-1}/X_{k-2}. All these spheres meet at exactly one point, where the crushed X_{k-2} collapsed: we say that X_{k-1}/X_{k-2} is a *bouquet of spheres.* The attaching map f of a k–cell to X_{k-1} gives rise to a map $\tilde{f} : \mathbb{S}^{k-1} \to X_{k-1}/X_{k-2}$, by composition with the quotient map. Hence we have a map $\tilde{f}$ from a sphere of dimension $k-1$ to a bouquet of spheres, all of dimension $k-1$.

Digression on degree and homotopy. Any continuous map from a sphere S_1 to a sphere S_2 of same dimension comes equipped with a *degree*, which, informally, is an integer which measures the number of times S_1 "wraps around" S_2 under this map. This integer can be negative, as we keep track of orientation. Since the proper topological definition of degree requires homology (which we are in the process of defining), we restrict ourselves to differentiable maps. The degree of a differentiable map f between two manifolds of same dimension is given by:

$$deg(f) = \sum_{\boldsymbol{x} \in f^{-1}(\boldsymbol{z})} 1 \cdot (sign \det Df_{\boldsymbol{x}}) \tag{61.1}$$

where $\boldsymbol{z}$ is any regular value of f, i.e. the determinants in the above sum are not zero (by Sard's theorem, almost all values of a smooth map are regular). It turns out that the above number is independent of the (regular) point $\boldsymbol{z}$. The degree of a map is invariant under homotopy of the map. [Two continuous maps f_0 and f_1 between the manifold M and the manifold N are *homotopic* if there is a continuous map $F : [0,1] \times M \to N$ such that $F(0,\boldsymbol{z}) = f_0(\boldsymbol{z}), \quad F(1,\boldsymbol{z}) = f_1(\boldsymbol{z})$ for all $\boldsymbol{z}$ in M.]

Back to horticulture. The attaching map $\tilde{f} : \mathbb{S}^{k-1} \to X_{k-1}/X_{k-2}$ has a multiple degree: on each sphere S_i in the bouquet one can compute the oriented number of preimages under $\tilde{f}$ of a regular point as in (61.1) (without loss of generality, we can assume that $\tilde{f}$ is differentiable except at the common point of the spheres). Suppose that $\boldsymbol{c}_1^{k-1}, \ldots, \boldsymbol{c}_{N_{k-1}}^{k-1}$ denote the $(k-1)$–cells of the cell complex and $\boldsymbol{c}_1^k, \ldots, \boldsymbol{c}_{N_k}^k$ its k-cells. We now form an $N_k \times N_{k-1}$ integer matrix ∂_k whose entry $\partial_k(ij)$ is the degree of the attaching map from $\partial \boldsymbol{c}_i^k$ to the jth sphere of the bouquet, i.e. $\boldsymbol{c}_j^{k-1}/\partial \boldsymbol{c}_j^{k-1}$. The matrices ∂_k, for $k \in \{1, \ldots, dimX\}$ essentially give all the combinatorial data describing how the complex X is pieced together from our collection of cells.

Chain complexes. We now want to view the matrices ∂_k as those of linear maps between finite dimensional vector spaces, or modules. To do this, one thinks of $\boldsymbol{c}_1^k, \ldots, \boldsymbol{c}_{N_k}^k$ as the basis vectors of an abstract vector space (or free module) C_k whose elements are formal sums of the form

$$\boldsymbol{c} = \sum_1^{N_k} a_j \boldsymbol{c}_j^k,$$

where a_j is an element of some "coefficient" field (or ring) K (usually $\mathbb{Z}_2, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$). Hence C_k is generated by the k-cells and $dim C_k = N_k$. For convenience, we define $\partial_0 \equiv 0$ on C_0.

Lemma 61.4

$$\partial_{k-1} \circ \partial_k \equiv 0. \tag{61.2}$$

(The proof of this crucial lemma, which we will not give here (see, *eg.* Dubrovin & al. (1987)) usually uses the long exact sequence of a triple and a pair in simplicial homology). A chain of maps and vector spaces (or modules):

$$C_n \overset{\partial_n}{\to} C_{n-1} \to \dots C_k \overset{\partial_k}{\to} C_{k-1} \to \dots \to C_0$$

satisfying (61.2) is called a *chain complex*.

Definition 61.5 The kth *homology group* of the finite cell complex X with coefficients in a ring K is given by:

$$H_k(X;K) = Ker\ \partial_k / Im\ \partial_{k+1}.$$

where, by convention, $\partial_0 = 0 = \partial_{n+1}$

This definition makes sense since, by Lemma 61.4, $Im\ \partial_{k+1} \subset Ker\ \partial_k$.

Note that $H_k(X) = 0$ whenever $k > \dim X$ or $k < 0$, since for such k, $X_k = \emptyset$.

Example 61.6 The circle $\mathbb{S}^1$ is a CW complex: we start with a point p and attach to it an interval $I = [0,1]$: the boundary points of I become identified to p under the attaching map. Using $\mathbb{R}$ as coefficients in our chain complex, we get $C_0 = \mathbb{R}.p \cong \mathbb{R}, C_1 = \mathbb{R}.I \cong \mathbb{R}$. The map $\partial_1 = 0$: p has the two preimages $\{0\}$ and $\{1\}$ under the attaching map, but they come with opposite orientation under the orientation induced by I.

Example 61.7 Figure 61. 2 gives the generators for a chain complex for the torus: $C_0 = \mathbb{R} \cdot \boldsymbol{z}, C_1 = \mathbb{R} \cdot a \oplus \mathbb{R} \cdot b, C_2 = \mathbb{R} \cdot D$. All the boundary maps are 0 in this case: $\partial_1 a = 0$ because it geometrically yield z twice but with opposite orientation. Likewise for $\partial_1 b$. As for $\partial_2 D = a + b - a - b = 0$, again due to orientation. Hence $Ker\ \partial_k = Im\ \partial_{k+1} = C_k$ for $k = 1, 2, 3$. We have shown:

$$H_k(\mathbb{T}^2, \mathbb{R}) \cong \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R}^2 & k = 1 \\ \mathbb{R} & k = 2. \end{cases}$$

Clearly, this result remains valid if we replace $\mathbb{R}$ by any coefficient ring K.

Example 61.8 A less trivial example is given by the Klein bottle. This non orientable surface is a torus with a twist and it cannot be embedded in $\mathbb{R}^3$. We build it with the same cells $\boldsymbol{z}, a, b$ and D as the torus. The only change occurs in the definition of ∂_2: instead of gluing D to two copies of b in opposite orientation, we give them the same orientation (see Figure 61.4). As a result, the matrix of ∂_2 is now $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Let us use the integers $\mathbb{Z}$ as our coefficient ring. Then $Ker\ \partial_2 = \{0\}$. From this we immediately get that $H_2(Klein, \mathbb{Z}) = 0$. As in the torus, $\partial_1 = 0$ so that $Ker\ \partial_1 = C_1 = a \cdot \mathbb{Z} \oplus b \cdot \mathbb{Z}$. Since $Im\ \partial_2 = \{0\} \cdot a \oplus 2\mathbb{Z} \cdot b$, $H_1(Klein, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. As in the case of the torus, $H_0(Klein, \mathbb{Z}) = \mathbb{Z}$ (in fact, the rank of H_0 gives the number of connected components of a manifold). Now let's reexamine the above computation with coefficients $K = \mathbb{Z}_2$ instead: the map $\partial_2 = 0$ in this case since 2=0 in this ring. Thus, in this case we are back to the same situation as with the torus: $H_0(Klein, \mathbb{Z}_2) \cong \mathbb{Z}_2$, $H_1(Klein, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $H_2(Klein, \mathbb{Z}_2) \cong \mathbb{Z}_2$. Finally, let's choose $K = \mathbb{R}$. Since $Ker\ \partial_2 = C_2$ in this case again, $H_2(Klein, \mathbb{R}) = \mathbb{R}$. Since $\mathbb{R}/2\mathbb{R} = \mathbb{R}/\mathbb{R} = \{0\}$, $H_1(Klein, \mathbb{R}) \cong \mathbb{R}$. As before $H_0(Klein, \mathbb{R}) \cong \mathbb{R}$.

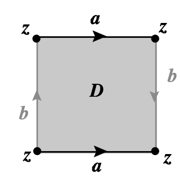

Fig. 61. 4. A cell decomposition for the Klein bottle. The only difference with that of the torus is the orientation of one of the segments b.

Some general properties and definitions related to homology. Let X be a compact manifold of dimension n. As we will see in next section, it can always be decomposed into a finite CW complex.

•$dimH_k(M, \mathbb{R}) = rank\ H_k(M, \mathbb{Z}) = b_k$ is the k^{th} *Betti number* of M.

•$\sum_{k=1}^{n}(-)^k b_k = \chi(M)$ is the *Euler characteristic* of M.

•Neither b_k nor $\chi(M)$ depend on the chain decomposition chosen for M.

•b_0 gives the number of connected component of M.

•$b_n = 1$ if M is orientable, $b_n = 0$ if M is not orientable.

Topological invariance. The importance of homology stems in great part from its invariance under topological equivalences. One topological equivalence is that of homeomorphism. A coarser equivalence (see Exercise 61.10) is that of homotopy type. Two topological spaces M and N have the same *homotopy type* if there are continuous maps $\phi : M \to N, \psi : N \to M$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are homotopic to the Identity map of M and N respectively. In other words M can be deformed into N and vice-versa.

Theorem 61.9 *If the two manifolds M and N are homeomorphic, or have the same homotopy type, then they have same homology: $H_*(M) = H_*(N)$ (the star $*$ stands for any integer).*

Since the degree of the attaching maps are invariant under homotopy, homology is itself an invariant under homotopy equivalence (this requires a more rigorous proof, of course, see *eg.* Dubrovin & al. (1987)).

C. Cohomology

Roughly speaking, cohomology is dual to homology. For readers of this book, it might be easier to see it through differential forms, which are dual to chains of cells in the sense that the integral $< c, \omega >= \int_c \omega$ of a form ω on a chain c is a linear, real valued function in c (it is also linear in ω). The duality bracket given by integration also satisfies:

$$< \partial\, c, \omega > = < c, d\omega >$$

where d is the exterior differentiation on forms. This formal equality is a general requirement for defining cohomology. In the case of forms it is simply given by Stokes' Theorem. Finally, we can define the cochain complex

$$C_0^* \xrightarrow{d_1} C_1^* \xrightarrow{d_2} \dots \xrightarrow{d_n} C_n^*$$

where $C_k^* = \Lambda^k$ is the vector space of k–forms and d_k is exterior differentiation. As with homology, we can define the *DeRham cohomology group* as:

$$H^k(M, \mathbb{R}) = Ker\ d_{k+1}/Im\ d_k,$$

$i.e.$ this cohomology is the quotient of closed forms over exact forms. One notable difference between homology and cohomology is the direction of the arrows in the complexes that defines them. Another notable difference, which makes the use of cohomology often preferable, is the existence of a natural product operation in cohomology, called the *cup product*. In DeRham cohomology, this cup product takes the form of wedge product of the forms:

$$[\omega_1] \cup [\omega_2] = [\omega_1 \wedge \omega_2]$$

where the notation $[\omega]$ denotes the class of the closed form ω. There are many different ways to define cohomology, but it can be shown that (given some normalization requirements), they all give the same result on compact manifolds. Poincaré, for instance, introduced cohomology (not under that name) by geometrically constructing a dual complex to a triangulation (a special CW chain decomposition). In the next section, where unstable manifolds of critical points of a Morse function will provide us with a chain decomposition, the dual decomposition can be taken to be that of stable manifolds.

D*. Covering Spaces and Fundamental Group

Covering spaces. The simple notation $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is rich in geometric and algebraic meaning. The quotient map $p : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is an instance of a *covering map* in that it is a local homeomorphism which is such that each point in the torus has an *evenly covered* neighbourhood U such that $p^{-1}U$ is made out of disjoint copies of U ($eg.$ take a disk of radius less than 1 U around the point z). That makes $\mathbb{R}^2$ a *covering space* of $\mathbb{T}^2$. In a covering space, the transformations that permute points in a fiber $p^{-1}(z)$ are homeomorphisms which form a group under composition called the group of *deck transformations*. For instance, $\mathbb{Z}^2$ is the group of deck transformations of the covering space $\mathbb{R}^2 \to \mathbb{T}^2$.

Lifting of curves. One can lift curves from a space M to its covering $\tilde{M}$ in a well prescribed way: if the curve γ starts at z_0 in M, choose one $\tilde{z}_0 \in p^{-1}(z_0) \subset \tilde{M}$ to start the *lift* of γ, $i.e.$ a curve $\tilde{\gamma}$ such that $p(\tilde{\gamma}) = \gamma$. Above an evenly covered neighbourhood U of z_0, there is only one way to define $\tilde{\gamma}$, since there is only one copy of U containing our choice $\tilde{z}_0$. One then proceed by continuity, covering γ with a finite number of overlapping evenly covered neighborhoods. A curve has as many distinct lifts as there are preimages of its starting point.

Classification of covering spaces for $\mathbb{T}^2$. We can construct other covering spaces of the torus, with other groups of deck transformations. For instance, the cylinder $\mathbb{R} \times \mathbb{S}^1 = \mathbb{R}^2/\mathbb{Z}$ is a covering space of the torus with deck transformations group $\mathbb{Z}$. $\mathbb{R}^2/(2\mathbb{Z} \oplus 3\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}^2$ is also a covering space which is itself a torus, but "6 times as big" as the standard one it covers. It has $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ as group of deck transformations, reflecting the finite number of elements a fiber $p^{-1}(z)$ has. In general, given any normal subgroup G of $\mathbb{Z}^2$, you get a covering space $\mathbb{R}^2/G \to \mathbb{T}^2$ with deck transformations group $\mathbb{Z}^2/G$. In fact these are *all* the possible covering spaces of the torus!

The fundamental group. The above classification generalizes to any manifolds, as we will see. Given a connected manifold M, we need to find a covering space which serves the role $\mathbb{R}^2$ does for $\mathbb{T}^2$. It turns out that the defining feature $\mathbb{R}^2$ has in this context is that it is connected and *simply connected*: any loop in $\mathbb{R}^2$ is homotopic to a point, or constant loop. This makes $\mathbb{R}^2$ the *universal cover* of $\mathbb{T}^2$: it is the unique (up to homeomorphism) covering space of $\mathbb{T}^2$ which is simply connected. Its uniqueness comes from a construction which works for any manifold. Choose some point z_0 in your manifold M. Declare that two curves starting at z_0 are equivalent if they have same endpoint and are homotopic. Define the covering space $\tilde{M}$ as the set of all such equivalence classes. If $[\gamma] \in \tilde{M}$ is one such equivalence class, define the covering map as $p([\gamma]) = \gamma(1)$ (its endpoint). One can indeed show that, with the appropriate topology, this is a covering space, and its deck transformations form a group called the *fundamental group* of M, denoted by $\pi_1(M, z_0)$ or $\pi_1(M)$ in short (changing the base point yields isomorphic groups). Since a deck transformation must permute points in a fiber, $\pi_1(M)$ is the group of all homotopy classes of loops based at a chosen point, with group law given by concatenation of two loops (*i.e.* follow one, then the next, which is possible since they have same endpoints). The inverse of a loop is the same loop traversed backwards. As an example, since $\mathbb{Z}^2$ is the deck transformation for the universal cover $\mathbb{R}^2$ of $\mathbb{T}^2$, we must have $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$. More generally $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$. On the other hand $\pi_1(\mathbb{S}^n) = \{0\}$, since the sphere is itself simply connected (and thus is its own universal cover).

Classification of covering spaces of any manifold M. As stated above, we can use the universal cover $\tilde{M}$ to classify all covering spaces of the (connected) manifold M: any other covering space N of M is of the form $N \cong \tilde{M}/G$ where G is a subgroup of $\pi_1(M)$. Furthermore, $G \cong \pi_1(N)$ and if G is a normal subgroup of $\pi_1(M)$, then the deck transformations of $N \to M$ form the group $\pi_1(M)/G$. Remember that G is normal if $aGa^{-1} = G$ for any $a \in \pi_1(M)$. As an example, any subgroup of $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$ is normal, since $\mathbb{Z}^2$ is abelian.

Fundamental group vs. homology. Note that $\pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2 \cong H_1(\mathbb{T}^2, \mathbb{Z})$. This is not a coincidence: both groups were constructed as equivalence classes based on closed loops. In general, a theorem of Poincaré (1895) says that $H_1(M, \mathbb{Z})$ is the abelianization of $\pi_1(M)$: it is the fundamental group made commutative. The way to abelianize a group G is by taking its quotient with the subgroup $[G, G]$ of its commutators, which are of the form $xyx^{-1}y^{-1}$. Hence we can write Poincaré's theorem as:

$$H_1(M, \mathbb{Z}) \cong \pi(M)/[\pi_1(M), \pi_1(M)].$$

To see how the case $M = \mathbb{T}^2$ fits here, note that $\mathbb{Z}^2$ is already abelian. In general, $\pi_1(M)$ can be much more complicated than $H_1(M, \mathbb{Z})$. Finally, this leads us to an important case of covering space, called *universal abelian cover* of a manifold M. It is the covering $\tilde{M}/[\pi_1(M), \pi_1(M)] \to M$ which, since the subgroup $[\pi_1(M), \pi_1(M)]$ is always normal, has (abelian) deck transformation group $H_1(M, \mathbb{Z}) \cong \pi_1(M)/[\pi_1(M), \pi_1(M)]$.

Exercise 61.10 Show that the circle and the cylinder have same homotopy type but are not homeomorphic.

Exercise 61.11 Using Exercise 61.3 compute the homology of $\mathbb{S}^n, \mathbb{T}^n, \mathbb{RP}^n$.

Exercise 61.12 Convince yourself, looking at Figure 61. 3 that the fundamental group of the double torus is $< a, b, c, d; aba^{-1}b^{-1}cdc^{-1}d^{-1} >$, *i.e.* the group generated by the elements a, b, c, d together with the relation that $aba^{-1}b{-}1cdc^{-1}d^{-1} = e$, the neutral element. What is the first homology group of the double torus? Repeat the question for surfaces of genus g.

62.* Morse Theory

We now show how any compact manifold can be described as a cellular space, with cells given by the unstable manifolds of the critical points of a Morse function. This immediately yields a relationship between critical points and Homology, in the guise of the Morse Inequalities. We first define some of these terms.

Let $f : M \to \mathbb{R}$ be a differentiable function on a manifold M. A *critical point* for f is a point $\boldsymbol{z}$ at which the differential of f is zero: $df(\boldsymbol{z}) = 0$. If f is twice differentiable, the critical point $\boldsymbol{z}$ is called *nondegenerate* if

$$\det \frac{\partial^2 f(\boldsymbol{z})}{\partial x^2} \neq 0 \tag{62.1}$$

where this second derivative is taken with respect to any local coordinates x around $\boldsymbol{z}$ on M. The function f is a *Morse function* if all its critical points are nondegenerate. One can show that there are many Morse functions on any manifold. In fact Morse functions are generic in the set of twice differentiable functions. See e.g. Guillemin & Pollack (1974) , as well as Milnor (1969) .

Note that the condition (62.1) is independent of the coordinate system. Indeed, at a critical point $\boldsymbol{z}$,

$$\frac{\partial^2 f(\boldsymbol{z})}{\partial y^2} = \frac{\partial x}{\partial y}^t \frac{\partial^2 f(\boldsymbol{z})}{\partial x^2} \frac{\partial y}{\partial x}.$$

This last formula also implies that the number of negative eigenvalues of the real, symmetric matrix $\frac{\partial^2 f(z)}{\partial x^2}$ does not depend on the coordinate system chosen around the critical point $\boldsymbol{z}$. This number is called the *Morse index* of $\boldsymbol{z}$. Qualitatively, the level set portrait of a function around a nondegenerate critical point is entirely determined by the index of the critical point. Indeed:

Lemma 62.1 (Morse Lemma) *Let $\boldsymbol{z}$ be a nondegenerate critical point for a function f on a manifold of dimension n. There is a coordinate system $\boldsymbol{x}$ around $\boldsymbol{z}$ such that:*

$$f(\boldsymbol{x}) = f(\boldsymbol{z}) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2.$$

We refer the reader to Milnor (1969) for a proof of this lemma, which generalizes the diagonalization process (Gram-Schmidt) for bilinear forms. Since the Morse Lemma clearly implies that the critical points of a Morse functions are isolated, we have:

Corollary 62.2 *A Morse function on a compact manifold has a finite number of critical points.*

The *gradient flow* of a function f is the solution flow for the O.D.E.:

$$\dot{z} = -\nabla f(z). \tag{62.2}$$

The *gradient* ∇f is defined here by $\langle \nabla f, . \rangle = df(.)$, where the brackets denotes some chosen Riemannian metric. The minus sign is put in (62.2) so that F decreases along the flow:

$$\frac{d}{dt} f(z(t)) = -\{\nabla f(z(t))\}^2 \leq 0$$

with equality occurring exactly at the critical points. The eigenvectors corresponding to the negative eigenvalues of $\frac{\partial^2 f(z)}{\partial x^2}$ span a subspace of $T_z M$ which is tangent to the unstable manifold at z of the gradient flow: that is, the $x_1, \ldots, x_k$ plane given by the Morse Lemma. We remind the reader that the *unstable manifold* of a restpoint for a flow is the manifold of points whose backward orbit is asymptotic to the restpoint. Hence *the Morse index of a nondegenerate critical point of a Morse function is the dimension of its unstable manifold.*

Remark 62.3 Note that if the metric chosen to define the gradient is the euclidean one in the Morse coordinate chart, the $(x_1, \ldots, x_k)$ plane is itself the unstable manifold of the critical point, at least in that chart. This can always be arranged, by a local perturbation of the metric, and we will assume from now on that this is the case.

The gist of Morse theory consists in studying how the topology of the *sublevel sets*:

$$M^a = \{x \in M \mid f(x) \leq a\}$$

changes as a varies.

Theorem 62.4 *If there is no critical points in $f^{-1}[a, b]$, then M^a and M^b are diffeomorphic. The inclusion of M^a in M^b is a deformation retraction.*

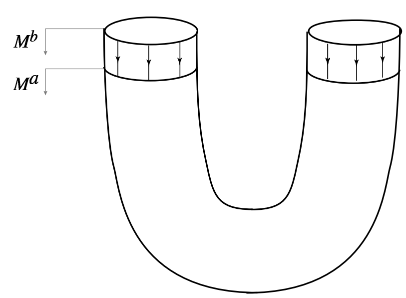

Fig. 62. 0. Deformation of a sublevel set M^b into the sublevelset M^a when there are no critical points in $f^{-1}[a, b]$. The lines with arrows represent trajectories of the gradient flow.

Proof. Deform M^b into M^a by flowing down the trajectories of the gradient flow, with appropriate speed and during an appropriate time interval. This is possible as long as there are no critical value in $[a, b]$. See Figure ??? □

Theorem 62.5 *Suppose $f^{-1}[a,b]$ is compact and has exactly one critical point in its interior, which is degenerate and of index k. Then M^b has the homotopy type of M^a with a cell of dimension k attached, namely, a ball in the unstable manifold of the critical point.*

Proof. (sketch) Let $\boldsymbol{z}$ be the critical point, $c = f(\boldsymbol{z})$ and $\epsilon > 0$ be a small real number. By the previous theorem, $M^{c+\epsilon}$ has the same homotopy type as M^b and likewise for $M^{c-\epsilon}$ and M^a. Hence, we just have to show that $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a cell attached.

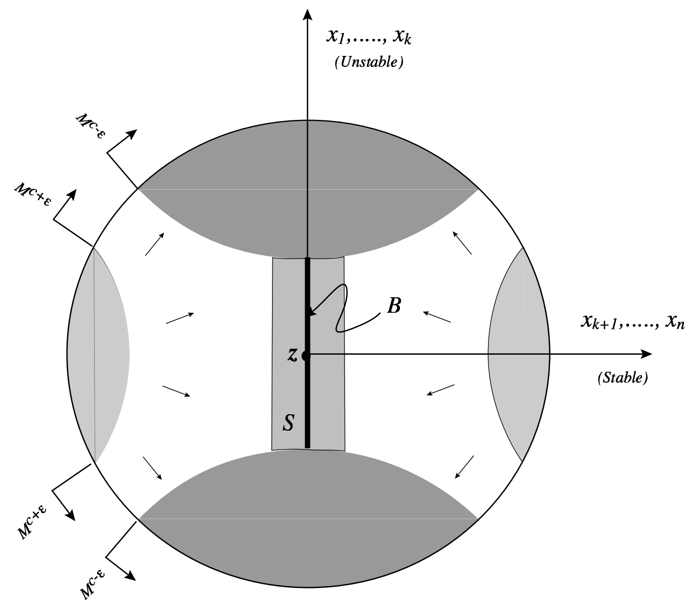

Fig. 62. 0. A neighborhood of a Morse critical point $\boldsymbol{z}$. A suitable parameterization of the flow retracts $M^{c+\epsilon}$ onto $M^{c-\epsilon} \cup S$, which itself can be deformed into $M^{c-\epsilon} \cup B$.

We have represented in Figure 62. 0 the sets $M^{c\pm\epsilon}$ within a Morse neighborhood. The drawing makes it intuitively clear that some reparameterization of the gradient flow (which we have represented by some arrows) will collapse $M^{c+\epsilon}$ into $M^{c-\epsilon} \cup S$. But the set S is given by:

$$S = \{f \leq c + \epsilon, x_1^2 + \ldots + x_k^2 \leq \delta\},$$

which can obviously be deformed into:

$$B = \{f \leq c + \epsilon, x_1 = \ldots = x_k = 0\},$$

that is, a ball in the unstable manifold of $\boldsymbol{z}$. In other words,

$$M^{c+\epsilon} \simeq M^{c-\epsilon} \cup B.$$

□

Any cellular space X is homotopically equivalent to a finite cell complex Y, where X and Y have the same number of cells in each dimension (one deforms each of the attaching maps defining X into one that attaches to cells of lower dimensions, see Dubrovin & al. (1987) , Section 4). This and the previous theorem yield:

Corollary 62.6 *Any sublevel set M^a of a Morse function on a compact manifold M has the homotopy type of a finite CW complex, whose cells correspond to the unstable manifolds of the critical points.*

Hence, since there always is a Morse function on any given manifold,

Corollary 62.7 *Any compact manifold has the homotopy type of a finite CW complex, with a cell of dimension k for each critical point of index k.*

Corollary 62.8 (Morse inequalities) *Given any Morse function f on a compact manifold M, the homology of M is generated by a finite complex $\{C_k, \partial_k\}_{\{1,\ldots,dim M\}}$ whose generators correspond to the critical points of index k of f. In particular, if $c_k = dim C_k$ is the number of critical points of index k,*

$$c_k \geq b_k = rank H_k(M, \mathbb{Z}) \tag{62.3}$$

and, better:

$$c_k - c_{k-1} + \ldots \pm c_0 \geq b_k - b_{k-1} + \ldots \pm b_0, \tag{62.4}$$

with equality holding for $k = n$.

Proof. The first statement in the theorem is somewhat of a tautology for us, since we have "defined" the homology of M as the cellular homology of any cellular complex representing M. Formula (62.3) is then trivial, since

$$H_k(M) = Ker\ \partial_k / Im\ \partial_{k+1},$$

and $Ker\ \partial_k$ is a subspace of C_k.The inequalities (62.4) are a consequence of (62.3) and their proof, left to the reader, only involves linear algebra. □

Remark 62.9 One can give a nice geometric interpretation of the maps ∂_k in the context of Morse theory. Assume that the gradient flow ϕ^t of our chosen Morse function is *Morse-Smale*, i.e. that for any given pair of critical points $\boldsymbol{x}, \boldsymbol{z}$, their respective stable and unstable manifold meet transversally. This is again a generic situation, which has the following implications: the set

$$M(\boldsymbol{x}, \boldsymbol{z}) = W^u(\boldsymbol{x}) \cap W^s(\boldsymbol{z}),$$

which is the union of all orbits connecting $\boldsymbol{x}$ and $\boldsymbol{z}$, is a manifold and

$$dim M(\boldsymbol{x}, \boldsymbol{z}) = index(\boldsymbol{x}) - index(\boldsymbol{z}).$$

In particular, if $index(\boldsymbol{x}) - index(\boldsymbol{z}) = 1$, $M(\boldsymbol{x}, \boldsymbol{z})$ is a one dimensional manifold made of a finite number of arcs that one can count, with $\pm$ according to a certain rule of intersection. This intersection number $m(\boldsymbol{x}, \boldsymbol{z})$ gives the coefficient in the generator $\boldsymbol{z}$ of $\partial(\boldsymbol{x})$, i.e.

$$\partial_k \boldsymbol{x} = \sum_{\boldsymbol{z} \in C_{k-1}} m(\boldsymbol{x}, \boldsymbol{z}).\boldsymbol{z}.$$

One can also define cohomology in this fashion: just take the same complex, but defined for the function $-f$. What was stable becomes unstable manifold and C_k becomes C_{n-k}. This not only gives us a geometric way to see cohomology, but a trivial proof of Poincaré's duality theorem:

$$H^{n-k}(M,\mathrm{I\!R}) \cong H_k(M,\mathrm{I\!R})$$

. For more details on this chain complex, which is sometimes called the Witten complex but dates back to J. Milnor's book on cobordism, see e.g. Salamon (1990) . For a proof of Poincaré's duality using the Morse complex, see Dubrovin & al. (1987) .

63. Controlling The Topology Of Invariant Sets

The relationship revealed by Morse between the critical point data of a function and the topology of the underlying manifold has a very wide generalization in the theory of Conley, which brings about a similar relationship for general continuous flows on locally compact topological spaces. We will outline this theory in Section 51.C. For now, we make a small step toward this generalization.

Here, and for the rest of this chapter, the cohomology used is the Čech cohomology with coefficients in $\mathrm{I\!R}$. We do not need to define this cohomology here: it is enough to state that it is well defined not only on manifolds but on their compact subsets as well. Furthermore it is continuous for the Hausdorff topology on compact subsets. Otherwise, it satisfies all the usual axioms and rules of cohomology and coincides with other cohomologies on compact manifolds.

Consider a compact set I which is invariant under the gradient flow of a function W on some finite dimensional manifold. If W is a Morse function, then necessarily I is made of critical points and the intersections of all their stable and unstable manifolds (prove it as an exercise!). Exactly as we did for manifolds, consider the Floer-Witten chain complex, generated by the critical points and with boundary maps given by the stable-unstable manifolds intersection data. It turns out (see the proof in Floer (1989) , and also Salamon (1990)) that this complex gives the (co)homology not of I, but of its Conley index, a topological/dynamical invariant of I that we define below. In certain cases, as in what follows, one can evaluate the Conley index and hence give lower estimates on the number of critical points. We use these results in Section 65 to estimate the number of critical points of functions on vector bundles.

Definition 63.1 Let M be a finite dimensional manifold. A compact neighborhood B in M is called an *isolating block* for a (continuous) flow ϕ^t if points on the boundary ∂B of B immediately leave B under the flow, in positive or negative time:

$$z \in \partial B \Rightarrow \phi^{(0,\epsilon)} \subset B^c \quad \text{or} \quad \phi^{(-\epsilon,0)} \subset B^c \quad \text{for some} \quad \epsilon = \epsilon(z) > 0.$$

The *exit set* B^- of B is defined as the set of points in ∂B which immediately flow out of B in *positive* time.

Given an isolating block B for the flow ϕ^t, define $I(B)$ to be the *maximal invariant set* included in B ("maximal" is in the sense of inclusion here). Alternatively:

$$I(B) = \cap_{t \in \mathbb{R}} \phi^t(B).$$

There are two classical ways to measure the topological complexity of an invariant set $I(B)$. One is its cohomology *cohomology Conley index*:

$$h(I) = H^*(B, B^-).$$

The bigger the dimension of this vector space, the more complex the topology of I. Note that in the notation $h(I)$, we have deliberately omitted the mention of B: this is because the vector spaces $H^*(B, B^-)$ are isomorphic for all isolating block B such that $I = I(B)$ (Conley & Zehnder (1984)). Hence $h(I)$ is an invariant of the set I. In practice, the size of $h(I)$ is measured by the *sum of the Betti numbers*

$$sb(h(I)) = \sum_k dim H^k(B, B^-).$$

This again is an invariant of I. A second, somewhat rougher way to measure the complexity of an invariant set I (or any topological space which admits continuous (semi)flows and a cohomology) is the *cuplength* which is defined as:

$$cl(I) = 1 + \sup\{k \in \mathbb{N} \mid \exists \omega_1, \ldots \omega_k, \quad \omega_j \in H^{n_j}(I), n_j > 1, \quad \text{and} \quad \omega_1 \cup \ldots \cup \omega_k \neq 0\}$$

The following is a generalization of both Morse and Lyusternick-Schnirelman theories. It is itself the consequence of the much more general theory of Conley for (semi)flows.

Theorem 63.2 *Let I be a compact isolated invariant set for the gradient flow of a function W on some manifold. If the function is Morse, the number of critical points in I is greater or equal to $sb(h(I))$. Otherwise, the number of critical points is at least equal to $cl(I)$.*

Historically, the first time Theorem 63.2 was applied in a significant way was in the proof of the following proposition, which appeared in several pieces in Conley & Zehnder (1983) :

Proposition 63.3 *Let M be a compact manifold and W be a real valued function on $M \times \mathbb{R}^n \times \mathbb{R}^m$. Suppose that the gradient flow of W admits an isolating block B of the form $B \simeq M \times D^+ \times D^-$ with exit set $M \times D^+ \times \partial D^-$, where $D^+ \subset \mathbb{R}^n, D^- \subset \mathbb{R}^m$ are homeomorphic to the unit balls. If W is a Morse function, it has at least $sb(M)$ critical points in B. In general, W has at least $cl(M)$ critical points.*

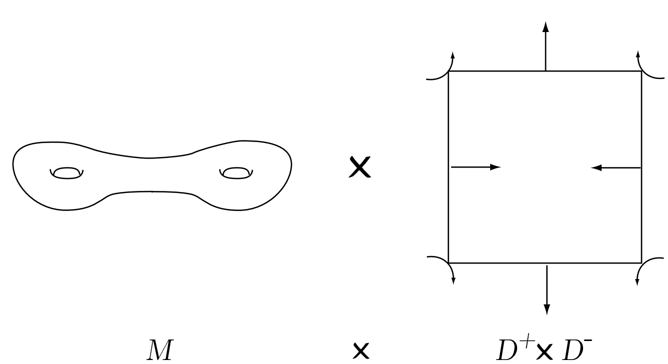

Fig. 63. 0. The isolating neighborhood in Proposition 63.3.

Conley and Zehnder applied this theorem in the case $M = \mathbb{T}^n$, where $sb(M) = 2^n$, and $cl(M) = n+1$, which gives a measure of the crudeness of the cuplength as compared to the sum of the Betti numbers. In the following section we will give a proof of Theorem 63.2 (we will only sketch the sum of betti number estimate, but give a complete proof of the cuplength estimate) as well as of Proposition 63.3.

64. Topological Proofs

The following lemma gives a situation where one can get a handle on the topology of an invariant set I. It is central to the proofs of several topological results we will use, including Proposition 63.3.

Lemma 64.1 (**Floer**) *Let B be an isolating block for a flow ϕ^t on a finite dimensional manifold, and I be its maximal invariant set. Suppose that there is a retraction $\alpha : B \to P$, where P is some compact subset of B. If there is a class $u \in H^*(B, B^-)$ such that :*

$$v \mapsto u \cup \alpha^*(v) : H^*(P) \to H^*(B, B^-)$$

is an isomorphism, then

$$\alpha_I{}^* : H^*(P) \to H^*(I)$$

is injective, where α_I denotes the restriction of α to I.

(If $N \subset M$ are two topological spaces and $i : N \to M$ is the inclusion map, a *retraction* is a map $r : M \to N$ such that $r \circ i = Id_N$, that is r restricts to Id on N). For a proof of Lemma 64.1, see Section 65.B.

Corollary 64.2 *Let B, I, P be as in Lemma 64.1, and let the flow ϕ^t in that lemma be the gradient of some function W. Then the number of critical points of W is at least $cl(P)$.*

Proof. If $H^*(P) \to H^*(I)$ is injective, $cl(I) \geq cl(P)$ and the Corollary is an immediate consequence of Proposition 63.2. □

A. Proof of the Cuplength Estimate in Theorem 63.2

Conley & Zehnder (1983) prove a cuplength estimate (their Theorem 5) that is valid for a compact invariant set I of a general flow ϕ^t. We follow their proof. Define a *Morse decomposition* for I to be a finite collection $\{M_p\}_{p\in P}$ of disjoint compact and invariant subsets of I, which can be ordered in such a way that any x not in $\cup_{p\in P} M_p$ is α-asymptotic to an M_j and ω-asymptotic to an M_i, with $i < j$ (x is *α-asymptotic* (resp. *ω-asymptotic*) to M_j if $\lim_{t\to -\infty\ (+\infty)} \phi^t(x) \in M_j$). One can show that a compact invariant set always has such a Morse decomposition. We now state Theorem 5 of Conley & Zehnder (1983) :

Theorem 64.6 *Let I be any compact invariant set for a continuous flow, and let $\{M_P\}_{p\in P}$ be a Morse decomposition for I. Then*

$$cl(I) \leq \sum_{p \in P} cl(M_p). \tag{64.1}$$

The relevant example for us is when ϕ^t is the gradient flow of a function with a finite number of (not necessarily nondegenerate) critical points on a compact invariant set I: it easy to check that these critical points form a Morse decomposition. Since an isolated point has trivial cohomology, $cl(M_p) = 1$ for each p in this example, and we have proven the cuplength estimate in Theorem 63.2. The case when the critical points are not isolated is trivial in that theorem: $cl(I) < \infty$ is always true... We now prove Theorem 64.6 .

Proof. Note that if $(M_1, \dots, M_k)$ is a Morse decomposition, then $(M_{1,\dots,k-1}, M_k)$ is also a Morse decomposition, where $M_{1,\dots,k-1}$ is formed by the union of $M_1 \cup \dots \cup M_{k-1}$ and of all the connecting orbits between these sets. Hence, by induction, we only need to consider the case where $k = 2$, and (M_1, M_2) is a Morse decomposition for I. From the definition of a Morse decomposition, we can deduce the existence of two compact neighborhoods I_1 of M_1 and I_2 of M_2 in I with $I_1 \cup I_2 = I$ and such that $M_1 = \cap_{t>0}\phi^t(I_1)$ and $M_2 = \cap_{t>0}\phi^t(I_2)$. In particular, by continuity of the Čech cohomology $H^*(I_j) = H^*(M_j), j = 1, 2$. Thus the proof of (64.1) reduces to that of the inequality $cl(I_1) + cl(I_2) \geq cl(I)$ whenever $I_1 \cup I_2 = I$ are three compact sets. The next lemma is devoid of dynamics:

Lemma 64.8 *Let $I_1 \cup I_2 \subset I$ be three compact sets. If $i_1 : I_1 \to I$, $i_2 : I_2 \to I$ and $i : I_1 \cup I_2 \to I$ are the inclusion maps, then, for any $\alpha, \beta \in H^*(I)$,*

$$i_1^*\alpha = 0 \quad \text{and} \quad i_2^*\beta = 0 \Rightarrow i^*(\alpha \cup \beta) = 0.$$

Proof. We chase the diagram:

$$\begin{array}{ccccc} H^*(I, I_1) & \otimes & H^*(I, I_2) & \overset{\cup}{\to} & H^*(I, I_1 \cup I_2) \\ \downarrow j_1^* & & \downarrow j_2^* & & \downarrow j^* \\ H^*(I) & \otimes & H^*(I) & \overset{\cup}{\to} & H^*(I) \\ \downarrow i_1^* & & \downarrow i_2^* & & \downarrow i^* \\ H^*(I_1) & \otimes & H^*(I_2) & \overset{\cup}{\to} & H^*(I_1 \cup I_2). \end{array}$$

The vertical sequences are exact sequences of pairs. Starting on the second line of the diagram with $\alpha, \beta \in H^*(I)$, suppose $i_1^*\alpha = 0 = i_2^*\beta$ then there must be $\tilde{\alpha} \in H^*(I, I_1)$ with $j_1^*\tilde{\alpha} = \alpha$, $\tilde{\beta} \in H^*(I, I_2)$ with $j_2^*\tilde{\beta} = \beta$. Now $j^*(\tilde{\alpha} \cup \tilde{\beta}) = \alpha \cup \beta$ and hence $i^*(\alpha \cup \beta) = i^* \circ j^*(\tilde{\alpha} \cup \tilde{\beta}) = 0$, by exactness. □

To finish the proof of Theorem 64.6, let $\alpha_1, \dots, \alpha_l$ be in $H^*(I)$ and $\alpha_1 \cup \dots \cup \alpha_l \neq 0$. Let this product be maximum, so that $cl(I) = l + 1$. Order the α's in such a way that $\alpha_1 \cup \dots \cup \alpha_r$ is the longest product not in the kernel of i_1^*. In particular $cl(I_1) \geq r + 1$ and $i_1^*(\alpha_1 \cup \dots \cup \alpha_r \cup \alpha_{r+1}) = 0$. Lemma 64.8 forces $i_2^*(\alpha_{r+1} \cup \dots \cup \alpha_l) \neq 0$ (i^* is one-to-one here, since $I_1 \cup I_2 = I$). Thus $cl(I_2) \geq l - (r + 1) + 1 = l - r$, and $cl(I_1) + cl(I_2) \geq l + 1 = cl(I)$. □

B. Proof of Lemma 64.1

In this subsection, we prove Lemma 64.1 that we restate here:

Lemma 64.1 **(Floer)** *Let B be an isolating block for a flow ϕ^t on a finite dimensional manifold, and I be its maximal invariant set. Suppose that there is a retraction $\alpha : B \to P$, where P is some compact subset of B. If there is a class $u \in H^*(B, B^-)$ such that :*

$$v \mapsto u \cup \alpha^*(v) : H^*(P) \to H^*(B, B^-)$$

is an isomorphism, then

$$\alpha_I{}^* : H^*(P) \to H^*(I)$$

is injective, where α_I denotes the restriction of α to I.

Proof. Define $B^\infty = \cap_{t>0} \phi^t B$, the set of points that stay in B for all negative time.

Lemma 64.9 *1) $H^*(B, B^\infty \cup B^-) = 0$*
2) $l^* : H^*(B^\infty) \to H^*(I(B))$ is an isomorphism, where $l : I(B) \to B^\infty$ is the inclusion.

Before proving this lemma, we use it to finish the proof of Lemma 64.1. Consider the diagram:

$$\begin{array}{ccccccc}
H^*(B,B^-) & \otimes & H^*(B,B^\infty) & \stackrel{\cup}{\to} & H^*(B,B^\infty \cup B^-) & = 0 \\
\downarrow Id & & \downarrow j^* & & \downarrow k^* & \\
H^*(B,B^-) & \otimes & H^*(B) & \stackrel{\cup}{\to} & H^*(B,B^-) & \\
& & \downarrow i^* & & & \\
& & H^*(B^\infty) & \stackrel{l^*}{\stackrel{\sim}{\to}} & H^*(I) &
\end{array}$$

where all vertical maps are induced by inclusions, and the two first horizontal maps are given by Künneth Formula. Suppose $\alpha_I{}^* v = 0$ for some $v \in H^*(P)$. Since l^* is an isomorphism and $\alpha_I = \alpha_{B^\infty} \circ l$, $0 = \alpha_I{}^* = l^*(\alpha_{B^\infty})^* v \Rightarrow (\alpha_{B^\infty})^* v = 0$. Since $\alpha_{B^\infty} = \alpha \circ i$, $0 = \alpha_{B^\infty}{}^* v = i^* \alpha^* v$. The middle, vertical sequence is the exact sequence of a pair. Hence there is a $w \in H^*(B, B^\infty)$ such that $j^* w = \alpha^* v$. But $u \cup \alpha^* v = k^*(u \cup w) = k^*(0) = 0$. The hypothesis of Lemma 64.1 forces $v = 0$. □

Proof of Lemma 64.9 Let $B^t = \phi^t(B)$ and $B^\infty = \cap_{t>0} B^t$ as before. Note in particular that, in the Hausdorff topology, $\lim_{t\to\infty} B^t = B^\infty$, and $\lim_{t\to 0} B^t = B$. To the triple of spaces $(B, B^- \cup B^t, B^-)$ corresponds the exact sequence:

$$\ldots \stackrel{\delta^*}{\to} H^*(B, B^t \cup B^-) \to H^*(B, B^-) \stackrel{i^*}{\to} H^*(B^t \cup B^-, B^-) \stackrel{\delta^*}{\to} H^{*-1}(B, B^t \cup B^-) \ldots,$$

(see *eg.* Dubrovin & al. (1987)). We now show that i^* is an isomorphism. Consider the diagram:

$$\begin{array}{ccc}
& & (B^t \cup B^-, B^-) \\
& \stackrel{i_1}{\nearrow} & \downarrow i \\
(B^t, B^- \cap B^t) & \stackrel{i_2}{\longrightarrow} & (B, B^-)
\end{array}$$

.

The excision theorem implies that i_1^* is an isomorphism, and the continuity of the Čech cohomology implies that i_2^* is an isomorphism. Since the diagram commutes, i^* must be an isomorphism. But this forces $H^*(B, B^t \cup B^-) = 0$ in the above diagram. Taking the limit of this equality as $t \to \infty$ proves 2).

Using the long exact sequence of the pair (B^∞, I), the map l^* induced by the inclusion $l : I \to B^\infty$ is an isomorphism whenever $H^*(B^\infty, I) = 0$, which we proceed to show. Note that $\phi^{-t}B^\infty \subset B^\infty$ and, by definition, $I = \cap_{t\geq 0}\phi^{-t}(B^\infty)$. Consider the maps:

$$(B^\infty, \phi^{-t}B^\infty) \stackrel{\phi^{-t}}{\to} (\phi^{-t}B^\infty, \phi^{-t}B^\infty) \stackrel{j}{\to} (B^\infty, \phi^{-t}B^\infty),$$

where j is the inclusion. The map $j \circ \phi^{-t}$ is clearly homotopic to Id, hence $H^*(B^\infty, \phi^{-t}B^\infty) \cong H^*(\phi^{-t}B^\infty, \phi^{-t}B^\infty) = 0$. Since this is true for all t, the continuity of the Čech cohomology concludes. □

C*. The Betti Number Estimate of Theorem 63.2 and Conley's Theory: a Sketch

We have proven in Theorem 64.6that, for a general function W, the number of critical points in an invariant set I for the gradient flow of W is greater than $cl(I)$. We now show that if W is a Morse function, the number of critical points in I is greater than $sb(I)$. To do so, one can either follow Floer (1989) in his generalization of the Witten complex (of unstable manifolds of critical points for gradient flows, see Remark 62.9) to invariant sets. His proof relies in part on Conley's theory. Alternatively, one can use Conley's generalized Morse inequalities that we state in this subsection.

Let I be a compact invariant set for a continuous flow ϕ^t and $(M_1, \ldots, M_k)$ be a Morse decomposition of I. Analogously to Theorem 64.6, Conley-Morse inequalities relate certain betti numbers of the Morse sets M_j to the corresponding betti numbers of I. To define the adequate betti numbers, we need to generalize the notion of isolating block to that of index pair for isolated invariant sets. A compact set I is an *isolated invariant set* if there is a neighborhood N of I such that $I = I(N)$ is the maximal invariant subset in N. An *index pair* for an isolated invariant set I is a pair of compact spaces (N_1, N_2) such that $N_1 \backslash N_2$ is a neighborhood of I and $I = I(N_1 \backslash N_2)$. This generalizes the concept of isolating block. In particular N_2 plays the role of the exit set, see Conley (1978), Conley & Zehnder (1984) . The fundamental property of these sets is that the homotopy type $[N_1/N_2, *]$ is independent of the choice of index pair for I and hence defines a topological invariant called the *Conley index* of the invariant set I. Giving less information, but easier to manipulate is the *cohomology Conley index* $H^*(N_1, N_2) = h(I)$, again an invariant of I. If $(N_1, N_2) = (B, B^-)$ for an isolating block B, this definition of $h(I)$ is the same as we have given previously. One way to encode the information given by $h(I)$ is via the coefficients of the *Poincaré polynomial*:

$$p(t, h(I)) := \sum_{j\geq 0} t^j dim H^j(N_1, N_2).$$

In Conley & Zehnder (1984) , it is proven that, given a Morse decomposition $(M_1, \ldots, M_k)$ for an invariant set I of a continuous flow ϕ^t, there is a *filtration* $N_0 \subset N_1 \subset \ldots \subset N_k$ such that (N_j, N_{j-1}) is an index pair for M_j. This is instrumental in proving the following:

Theorem 64.12 (Conley-Morse inequalities)

$$\sum_{j=1}^{k} p(t, h(M_j)) = p(t, h(I)) + (1+t)Q(t), \tag{64.2}$$

where $Q(t)$ is a polynomial with positive coefficients

This theorem is an extraordinary generalization of the classical Morse inequalities: it is valid for any *continuous* flow on a locally compact space (not necessarily a manifold!). To see that one indeed retrieves the betti number estimates of Theorem 63.2, one uses the Morse decomposition of our invariant set I given by the (isolated) critical points $z_1, \ldots, z_N$. Thanks to the Morse Lemma, it is not hard to construct an isolating block for each z_j, and show that the Conley index of z_j is a pointed sphere made by collapsing the boundary local unstable manifold of z_j to a point: take the set S in Figure 64. 1.

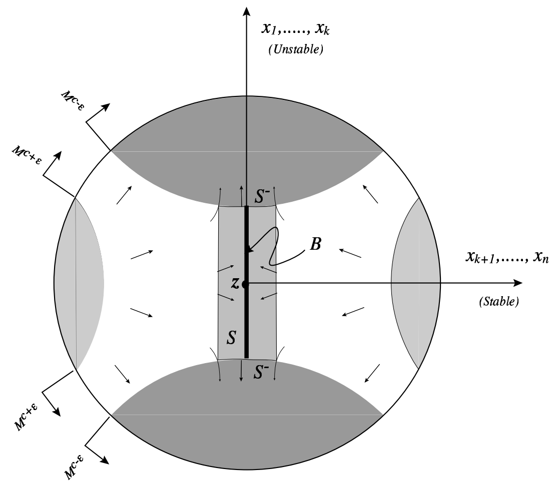

Fig. 64. 1. The index pair (S, S^-) retracts on (B, B^-), a pair made of the local unstable manifold of z and its boundary (a disk of dimension k equal to the index of the critical point z and its bounding sphere). Thus $h(z) = H^*(S, S^-) \cong H^*(B, B^-) \cong H^*(\$^k, *)$ which has exactly one generator in dimension k.

Hence $p(t, h(z_j)) = t^{u_j}$, where u_j is the Morse index of z_j. Now the pair $(I, \emptyset)$ is an isolating pair for I (no points exit I), and thus $p(t, h(I)) = \sum t^k dim H^k(I)$. The positivity of the coefficients of Q in (64.2) therefore insures that there are at least $dim H^k(I)$ critical points of index k.

D. Proof of Proposition 63.3

To prove this proposition, we let the manifold M play the role of P in Lemma 64.1. The retraction α of that lemma is given by the canonical projection $\alpha : B \to M$. Clearly the projection of B onto $M \times D^-$ is a deformation retract, which deforms B^- onto $M \times \partial D^-$. Hence $H^*(B, B^-) \cong H^*(M \times D^-, M \times \partial D^-)$. Now, Künneth Formula gives an isomorphism:

$$H^*(M) \otimes H^*(D^-, \partial D^-) \overset{\cup}{\overset{\sim}{\to}} H^*(M \times D^-, M \times \partial D^-)$$

where, as suggested by the notation, one gets all of the classes in the right hand side vector space as cup products of classes in the two left hand side spaces (with the appropriate identifications given by the inclusion maps). But, letting $n = dim D^-$, we have $H^*(D^-, \partial D^-) \cong H^*(\$^n, *)$, which has exactly one generator u in dimension n.

Hence $H^*(M) \cong H^{*+dim\ M}(B, B^-)$ and $sb(M) = sb(h(I))$ where I is the maximal invariant set in B. This and Theorem 63.2 yield the Betti number estimate. The homeomorphism $H^*(M) \cong H^*(B, B^-)$ is of the type prescribed by Lemma 64.1. This implies that the induced map $H^*(M) \to H^*(I)$ is injective and hence $cl(I) \geq cl(M)$. This fact and Theorem 63.2 give the cuplength estimate. □

E*. Floer's Theorem of Global Continuation of Hyperbolic Invariant Sets.

Floer's Lemma 64.1 is the cornerstone to the proof of the following theorem, where he makes good use of the powerful property of "invariance under continuation" of the Conley Index. This theorem illustrates the power of Conley's theory, and shows the historical root of Floer's Cohomology. Note that, in the theory of dynamical systems, the hyperbolicity of an invariant set for a dynamical system is intimately related to its persistence under *small* perturbations of the system: this relationship is the core of many theorems on structural stability. What is interesting about the following theorem (and Conley's theory in general) is that it provides situations when the persistence of an invariant set can be made global (but rough).

The notion of continuation of invariant sets makes use of the simple following fact: an index pair for a flow ϕ^t will remain an index pair for all flows that are C^0 close to ϕ^t. Two isolated invariant sets for two different flows are *related by continuation* if there is a curve of flows joining them (*i.e.* an isotopy) which can be (finitely) covered by intervals of flows having the same index pair. The following theorem (Theorem 2 in Floer (refine)) can be seen as an instance of weak, but *global*, stability of normally hyperbolic invariant sets.

Theorem 64.16 **(Floer)** *Let ϕ^t_λ be a one parameter family of flows on a C^2 manifold M. Suppose that G_0 is a compact C^2 submanifold invariant under the flow ϕ^t_0. Assume moreover that G_0 is* **normally hyperbolic**, *i.e. there is a decomposition:*

$$TM\big|_{G_0} = TG_0 \oplus E^+ \oplus E^-$$

which is invariant under the covariant linearization of the vector field V_0 corresponding to ϕ^t_0 with respect to some metric $\langle\ ,\ \rangle$, so that for some constant $m > 0$:

$$\begin{aligned} &\langle \xi, DV_0\xi\rangle \leq -m\langle \xi,\xi\rangle \text{ for } \xi \in E^- \\ &\langle \xi, DV_0\xi\rangle \geq m\langle \xi,\xi\rangle \text{ for } \xi \in E^+ \end{aligned} \tag{64.3}$$

Suppose that there is a retraction $\alpha : M \to G_0$ and that there is a family G_λ of invariant sets for ϕ^t_λ which are related by continuation to G_0 . Then the map:

$$\left(\alpha\big|_{G_\lambda}\right)^* : H^*(G_0) \to H^*(G_\lambda)$$

in Čech cohomology is injective.

In this precise sense, normally hyperbolic invariant sets continue *globally*: their topology can only get more complicated as the parameter varies away from 0.

65. Generating Phases Quadratic at Infinity

A. Generating Phases on Product Spaces

The following proposition serves a key role in various proofs in this book, as well as in symplectic topology.

Proposition 65.1 *Let M be a compact manifold, and W a real–valued function on $M \times \mathbb{R}^K$ satisfying:*

$$\lim_{\|\boldsymbol{v}\| \to \infty} \frac{1}{\|\boldsymbol{v}\|} \left(\frac{\partial W}{\partial \boldsymbol{v}}(\boldsymbol{q}, \boldsymbol{v}) - d\mathcal{Q}(\boldsymbol{v}) \right) = 0, \tag{65.1}$$

where $\mathcal{Q}(\boldsymbol{v})$ is a nondegenerate quadratic form on $\mathbb{R}^K$. Then W has at least $cl(M)$ critical points. If W is a Morse function, then it has at least $sb(M)$ critical points.

The function W of Proposition 65.1 is a special case of a class of function called *generating phases*. We develop this notion in the next subsection.

Proof. In an appropriate orthonormal basis $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_K)$ of $\mathbb{R}^K$,

$$\mathcal{Q}(\boldsymbol{v}) = \langle A\boldsymbol{v}, \boldsymbol{v} \rangle \quad \text{with} \quad A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_K \end{pmatrix},$$

with $a_j \neq 0$. Let $(v_1, \ldots, v_K)$ be the coordinates of an element $\boldsymbol{v} \in \mathbb{R}^K$ in this basis. We claim that:

$$B(C) = \{(\boldsymbol{q}, \boldsymbol{v}) \in M \times \mathbb{R}^K \mid \sup_j |v_j| \leq C\}$$

is an isolating block for the gradient flow of $-W$, when C is large enough.

To prove this, note that $B(C)$ is a compact neighborhood. Thus, in order to show that the flow exits in small positive or negative time at the boundary of $B(C)$, it suffices to check that, on each "face" $\{v_j = C\}$ of $\partial B(C)$, the dot product of ∇W with the normal vector to this face is non zero. The same argument will apply to the face $\{v_j = -C\}$. The normal unit vector pointing out at a point $\boldsymbol{z} = (\boldsymbol{q}, \boldsymbol{v})$ of $\{v_j = C\}$ is $\boldsymbol{e}_j$. But:

$$\begin{aligned} \frac{\partial W}{\partial \boldsymbol{v}}(\boldsymbol{z}).\boldsymbol{e}_j &= \frac{\partial \mathcal{Q}}{\partial \boldsymbol{v}}.\boldsymbol{e}_j + \left(\frac{\partial W}{\partial \boldsymbol{v}} - \frac{\partial \mathcal{Q}}{\partial \boldsymbol{v}} \right).\boldsymbol{e}_j \\ &= C\left(a_j + \frac{1}{C}\left(\frac{\partial W}{\partial \boldsymbol{v}} - \frac{\partial \mathcal{Q}}{\partial \boldsymbol{v}} \right).\boldsymbol{e}_j \right) \end{aligned}$$

This last expression must be of the sign of a_j, for large C as the last term inside the bracket tends to zero when $C \to \infty$ ($\|\boldsymbol{v}\|$ is of the order of C.) The same proof works for the face $\{v_j = -C\}$, since the outward normal vector is $-\boldsymbol{e}_j$ on this face. We have proved that, for all C larger than some C_0, the set $B(C)$ is an isolating block. Denote by B^- the *exit set* of $B = B(C)$, *i.e.* the subset of ∂B on which points flow out in positive time. In this case B^- is the union of the faces $\{v_j = \pm C\}$ such that the corresponding eigenvalue a_j is negative (remember, we are looking at the gradient flow of $-W$).

Hence $B \cong M \times D^+ \oplus D^-$ where the disks D^+, D^- are respectively the intersections of the positive and negative eigenspaces of $\mathcal{Q}$ with the set $\{\sup_j |v_j| \leq C\}$, and the exit set is $B^- \cong M \times D^+ \oplus \partial D^-$. We are exactly in the situation of Proposition 63.3 which gives us the appropriate estimates for the number of critical points inside B. □

B. Generating Phases on Vector Bundles

Proposition 65.1 is a cornerstone in the theory of generating phases. We now develop this theory a little and prove a generalization of Proposition 65.1 for functions on non trivial bundles which we will need. In Chapter 9, we will show how this theory gives an approach to symplectic topology.

Definition 65.2 A *generating phase* is a function

$$W : E \to \mathbb{R}$$

where E is the total space of a vector bundle $E \to M$ and M a manifold.

If moreover W satisfies:

$$\lim_{\|\boldsymbol{v}\|\to\infty} \frac{1}{\|\boldsymbol{v}\|}\left(\frac{\partial}{\partial \boldsymbol{v}}(W - \mathcal{Q})\right) = 0, \tag{65.2}$$

where, for each $\boldsymbol{q}$, $\mathcal{Q}(\boldsymbol{q}, \boldsymbol{v})$ is a nondegenerate quadratic form with respect to the fiber $\boldsymbol{v}$, then we say W is a *generating phase quadratic at infinity* , abbreviated *g.p.q.i.*.

We will see in Chapter 9 that the term "generating" refers to the fact that, provided they satisfy a generic condition in their derivative, generating phases generate Lagrangian manifolds of T^*M. Generating phases are also called *generating functions* when associated to the Lagrangian manifold that they generate, or *generating phase function*. We will show in Chapter 9 that twist maps generating functions *are* generating functions in this sense. We now define some elementary operations on generating phases. These will enable us to extend Proposition 65.1 to cover general g.p.q.i.'s. These operations are specially important in symplectic topology in that they enable one to define symplectic invariants of Lagrangian manifolds (capacities) as minimax values of their generating functions (see Viterbo (1992) and Siburg (1995)).

Definition 65.3 Let $W_1 : E_1 \to \mathbb{R}$, and $W_2 : E_2 \to \mathbb{R}$ be two generating phases. We say that W_1 and W_2 are *equivalent* if there is a fiber preserving diffeomorphism $\Phi : E_1 \to E_2$ such that:

$$W_2 \circ \Phi = W_1 + cst.$$

Definition 65.4 Let $W_1 : E_1 \to \mathbb{R}$ be a g.p.q.i. and $f : E_2 \to \mathbb{R}$ a nondegenerate quadratic form in the fibers of E_2. The function $W_2 : E_1 \oplus E_2 \to \mathbb{R}$ defined by:

$$W_2(\boldsymbol{q}, \boldsymbol{v}_1, \boldsymbol{v}_2) = W_1(\boldsymbol{q}, \boldsymbol{v}_1) + f(\boldsymbol{q}, \boldsymbol{v}_2)$$

is called a *stabilization* of W_1.

Proposition 65.5 *If the generating phase W_1 is equivalent to W_2, or is a stabilization of W_2 (or both) then critical points of W_1 are mapped bijectively into those of W_2 and the set of critical values are the same, up to a shift by a constant.*

Proof. Let $W_1 \circ \Phi = W_2 + C$ as in Definition 65.2. Then,

$$dW_1 = \Phi^* dW_2$$

Hence the set of critical points of W_1 is sent bijectively to that of W_2 by Φ. There is a constant discrepancy of C between critical values of W_1 and W_2 in this case .

Now let

$$W_2(\boldsymbol{q}, \boldsymbol{v}_1, \boldsymbol{v}_2) = W_1(\boldsymbol{q}, \boldsymbol{v}_1) + f(\boldsymbol{q}, \boldsymbol{v}_2)$$

be as in Definition 65.3. Critical points of a generating phases W satisfy, in particular, $\partial W/\partial \boldsymbol{v} = 0$. But here,

$$\frac{\partial W_2}{\partial \boldsymbol{v}} = (\partial W_1 \boldsymbol{v}_1, \partial f \boldsymbol{v}_2) = 0 \Rightarrow \boldsymbol{v}_2 = 0$$

and since any point $(\boldsymbol{q}, 0)$ of E_2 is critical for f, the critical points of W_2 correspond exactly to those of W_1. It is easy to see that the critical values of W_1 and W_2 are the same at the corresponding critical points. □

Proposition 65.6 *Let M be a compact manifold and $W : E \to \mathbb{R}$ be a g.p.q.i. on a fiber bundle $E \to M$. Then W has at least $cl(M)$ critical points. If W is a Morse function, then it has at least $sb(M)$ critical points.*

Proof. It is a corollary of Proposition 65.1 and of the following:

Lemma 65.7 *Let $W : E \to \mathbb{R}$ be a g.p.q.i. Then it is equivalent, after stabilization, to a g.p.q.i. $\overline{W} : M \times \mathbb{R}^K \to \mathbb{R}$ whose quadratic part $\overline{\mathcal{Q}}$ is independent of the base point.*

Proof. (We follow Theret (1999)) There exists a fiber bundle F such that $E \oplus F$ is trivial (*eg.* take F to be the dual of E, see Klingenberg (1982)). Stabilize W by endowing F with a nondegenerate quadratic form $\mathcal{Q}_2$. Since $E \oplus F$ is trivial, there is a fiber bundle diffeomorphism $\Phi : E \oplus F \to M \times \mathbb{R}^K$. A fiber bundle diffeomorphism being linear in each fiber, $(W \oplus \mathcal{Q}_2) \circ \Phi^{-1}$ is a g.p.q.i. on $M \times \mathbb{R}^K$.

We now show that any g.p.q.i. $W(\boldsymbol{q}, \boldsymbol{v})$ on a trivial bundle $M \times \mathbb{R}^K$ is equivalent to one with a quadratic part which is independent of the base point $\boldsymbol{q}$. Let $\mathcal{Q}$ be the quadratic part of W and write $\mathcal{Q}(\boldsymbol{q}, \boldsymbol{v}) = \langle A(\boldsymbol{q})\boldsymbol{v}, \boldsymbol{v}\rangle$, where $\langle\, , \rangle$ denotes the dot product on $\mathbb{R}^K$. Let $E^+_{\boldsymbol{q}} \oplus E^-_{\boldsymbol{q}} = E_{\boldsymbol{q}}$ be the decomposition of $E_{\boldsymbol{q}}$ into the positive and negative eigenspaces of $A(\boldsymbol{q})$. If the fiber bundles E^+ and E^- were trivial, the Gram-Schmidt diagonalization process would make $\mathcal{Q}$ equivalent to a constant quadratic form with matrix $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ on $E^+ \oplus E^-$, and the resulting fiber bundle diffeomorphism would make W equivalent to a g.p.q.i. such as we advertised. To arrive to this situation, stabilize $\mathcal{Q}\big|_{E^+}$ (*resp.* $\mathcal{Q}\big|_{E^-}$) to a positive definite $\mathcal{Q}^+$ (*resp.* negative definite $\mathcal{Q}^-$) on a *trivial* bundle $\mathcal{E}^+$ (*resp.* $\mathcal{E}^-$). □

Remark 65.8 Our definition of g.p.q.i. is more general than the one commonly found in the (french) literature (*i.e.* Sikorav (1986), Laudenbach & Sikorav (1985), Chaperon (1989), Theret (1999), Viterbo (1992)). Usually one asks that W *be equal* to its quadratic part $\mathcal{Q}$ outside of a compact set. One can show (see Theret (1999)) that if $W - \mathcal{Q}$ is bounded outside of a compact set, then W is equivalent, after stabilization, to such a g.p.q.i.. It is not clear to us that the same would hold with our more general asymptotic condition. In that sense, Proposition 65.6is stronger of its kind than any we know of in the literature.

Proposition gpqi or TOPOpropgpqi is 65.6, Proposition TOPOproptrivialgpqi is 65.1Theorem floerthm or TOPOthmfloer is 64.16, Proposition TOPOpropcz is 63.3, Section TOPOsectionproofs is 64, Section TOPOsectioninvtset is 63, Lemma TOPOlemfloer is 64.1, Theorem TOPOthmsbcl is 63.2, Section TOPOsecgpqi is 65