Appendix 1 or SG

OVERVIEW OF SYMPLECTIC GEOMETRY

(Oct 17 1999)

Action to be taken: Correct typos, decide whether to make this chapter an appendix or set it in the middle of the book (Jan 13 1999). Something weird in the header of last page!

Symplectic geometry is the language underlying the theory of Hamiltonian systems. This chapter is a short review of the main concepts, especially as they apply to Hamiltonian systems and symplectic maps in cotangent bundles. These spaces are natural when considering mechanical systems, where the base, or *configuration space* describes the position and the momentum belongs to the fiber of the cotangent bundle of the configuration space. In our optic of symplectic twist maps , one important concept studied in this chapter is that of exact symplectic map. Theorem SGhamexactsymp proves that Hamiltonian systems give rise to exact symplectic maps. We assume here some familiarity with the notions of manifold, vector bundles and differential forms. The reader who is uncomfortable with these concepts should consult any of the following references :Guillemin & Pollack (1974) or Spivak (1970) . For more on symplectic geometry and Hamiltonian systems, see Arnold (1978), Weinstein (1979), Abraham & Marsden (1985) or McDuff & Salamon (1996).

55. Symplectic Vector Spaces

In this section, we review some essentials of the linear theory of symplectic vector spaces and transformations. They will be our tools in understanding the infinitesimal behavior of symplectic maps and Hamiltonian systems in cotangent bundles. A *symplectic form* on a real vector space V is a bilinear form Ω which is skew symmetric and nondegenerate:

$$\begin{aligned} &\Omega(a\boldsymbol{v} + b\boldsymbol{v}', \boldsymbol{w}) = a\Omega(\boldsymbol{u}, \boldsymbol{w}) + b\Omega(\boldsymbol{u}', \boldsymbol{w}), \ \ (\boldsymbol{u}, \boldsymbol{u}', \boldsymbol{w} \in V, \ a, b \in \mathbb{R}). \\ &\Omega(\boldsymbol{u}, \boldsymbol{w}) = -\Omega(\boldsymbol{w}, \boldsymbol{u}) \\ &\boldsymbol{u} \neq 0 \Rightarrow \exists \boldsymbol{w} \text{ such that } \ \Omega(\boldsymbol{u}, \boldsymbol{w}) \neq 0 \end{aligned}$$

A *symplectic vector space* is a vector space V together with a symplectic form.

Example 55.1 The determinant in $\mathbb{R}^2$ is a symplectic form. More generally, the *canonical symplectic form* on $\mathbb{R}2n$, is given by:

$$\Omega_0(\boldsymbol{u}, \boldsymbol{w}) = \langle J\boldsymbol{u}, \boldsymbol{w} \rangle, \quad J = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$$

where the brackets $\langle\, , \,\rangle$ denote the usual dot product. We will see that all symplectic vector spaces "look" like this, in particular, their dimension is always even. Usually, one denotes:

$$\Omega_0 = d\boldsymbol{p} \wedge d\boldsymbol{q} = \sum_{k=1}^{n} dp_k \wedge dq_k$$

where it is understood that dq_k, dp_k are elements of the dual basis for the coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ of $\mathbb{R}2n$. The symplectic space $(\mathbb{R}2n, \Omega_0)$ can also be interpreted as $\mathbb{R}^n \oplus (\mathbb{R}^n)^*$, equipped with the canonical symplectic form:

$$\Omega_0(\boldsymbol{a} \oplus \boldsymbol{b}, \boldsymbol{c} \oplus \boldsymbol{d}) = \boldsymbol{d}(\boldsymbol{a}) - \boldsymbol{b}(\boldsymbol{c}).$$

It is often convenient to view a bilinear form as a matrix. To do this, fix a basis $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ of V, and set:

$$A_{ij}^{\Omega} = \Omega(\boldsymbol{e}_i, \boldsymbol{e}_j)$$

Equivalently, if $\langle\, , \,\rangle$ is the dot product associated with the basis $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$, then A^{Ω} is the matrix satisfying:

$$\Omega(\boldsymbol{u}, \boldsymbol{w}) = \langle A^{\Omega} \boldsymbol{u}, \boldsymbol{w} \rangle.$$

We now show that all symplectic vector spaces are isomorphic to the canonical $(\mathbb{R}2n, \Omega_0)$.

Theorem 55.2 (Linear Darboux) *If (V, Ω) is a symplectic space, one can find a basis for V in which the matrix A^{Ω} of Ω is given by* $A^{\Omega} = J = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$.

Hence, the isomorphism that sends each vector in V to its coordinate vector in the basis given by the theorem will be an isomorphism between (V, Ω) and $(\mathbb{R}^{2n}, \Omega_0)$. In classical notation, the coordinates in the Darboux coordinates are denoted by[15]

$$(\boldsymbol{q}, \boldsymbol{p}) = (q_1, \dots, q_n, p_1, \dots, p_n).$$

Proof. Since Ω is non degenerate, given any $\boldsymbol{v} \neq 0 \in V$, we can find a vector $\boldsymbol{w} \in W$ such that $\Omega(\boldsymbol{v}, \boldsymbol{w}) = -1$. In particular, the plane P spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$ is a symplectic plane and the bilinear form induced by Ω on P with this basis has matrix:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{55.1}$$

Since Ω is nondegenerate on P, we must have $P^{\perp} \cap P = \{0\}$. Furthermore $V = P + P^{\perp}$, since if $\boldsymbol{u} \in V$,

$$\boldsymbol{u} - \Omega(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{w} + \Omega(\boldsymbol{u}, \boldsymbol{w})\boldsymbol{v} \in P^{\perp}.$$

Ω must be nondegenerate on the $dimV - 2$ dimensional subspace $P^{\perp}$, so we can proceed by induction, and decompose $P^{\perp}$ into Ω–orthogonal planes on which the matrix of Ω is as in (55.1) . A permutation of the vectors of the basis we have found gives $A^{\Omega} = J$. □

[15] In the litterature, one also sees frequently $(\boldsymbol{p}, \boldsymbol{q})$, with $-J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$ as canonical matrix.

With any bilinear form Ω on a vector space comes a notion of *orthogonal subspace* $W^\perp$ to a given subspace (or vector) W :

$$W^\perp = \{\boldsymbol{u} \in V \mid \Omega(\boldsymbol{u}, \boldsymbol{w}) = 0, \forall \boldsymbol{w} \in W\}$$

In the case of symplectic forms, the analogy with the usual notion of orthogonality can be quite misleading, as a subspace and its orthogonal will often intersect.

Exercise 55.3 Show that the linear transformation whose matrix is J in the cannonical basis is orthogonal (i.e belongs to $O(2n)$), that it satisfies $J^2 = -Id$ (i.e. J is a complex structure) as well as

$$\Omega_0(Jv, Jw) = \Omega_0(v, w)$$

(that is, J is *symplectic*, see section 57.)

Exercise 55.4 Show that a one dimensional vector subspace in a symplectic vector space is included in its own orthogonal subspace.

Exercise 55.5 Show that in a Darboux basis for a symplectic plane,

$$\Omega(\boldsymbol{v}, \boldsymbol{w}) = det(\boldsymbol{v}, \boldsymbol{w}).$$

If (q_1, p_1) are the corresponding coordinates for the plane in this basis, this determinant form is denoted by $q_1 \wedge p_1$. Show that, in Darboux coordinates for a symplectic space of dimension $2n$,

$$\Omega = \boldsymbol{q} \wedge \boldsymbol{p} = \sum_1^n q_k \wedge p_k$$

Exercise 55.6 Prove that a general skew symmetric form Ω has "normal form":

$$A^\Omega = \begin{pmatrix} 0_k & -Id_k & \\ Id_k & 0_k & \\ & & 0_l \end{pmatrix}$$

where k, l do not depend on the basis chosen.

56. Subspaces of a Symplectic Vector Space

Let V be a symplectic vector space of dimension $2n$, $W \subset V$ a subspace, and Ω_W the symplectic form restricted to W. The previous exercise shows that we can find a basis for W in which :

$$A^{\Omega_W} = \begin{pmatrix} 0_k & -Id_k & \\ Id_k & 0_k & \\ & & 0_l \end{pmatrix} \quad dimW = 2k + l$$

In other words, (W, Ω_W) is determined up to isomorphism by k and its dimension. We will say that W is:

- *null* or *isotropic* if $k = 0$ (and $l = dimW$),
- *coisotropic* if $k + l = n$.
- *Lagrangian* if $k = 0$ and $l = n$. (i.e. W is isotropic and coisotropic.)
- *symplectic* if $l = 0$ and $k \neq 0$.

The *rank* of W is the integer $2k$.

The next theorem tells us that the qualitatively different subspaces of a symplectic space can be represented by coordinate subspaces in some Darboux coordinates.

Theorem 56.1 *A subspace W of rank $2k$ and dimension $2k+l$ in a symplectic space can be represented, in appropriate Darboux coordinates, by the coordinate plane:*

$$(q_1, \ldots, q_{k+l}, p_1, \ldots, p_k).$$

In particular, in some well chosen bases, an isotropic space is made entirely of q's and a coisotropic one must have at least n q's (the role of p's and q's can be reversed, of course) and a symplectic space has the same number of q's and p's.

Proof. From the definition of the rank of W, there is a subspace U of W of dimension $2k$ which is symplectic, on which we can put Darboux coordinates. $U^\perp \cap W$, the null space of Ω_W, is in the subspace $U^\perp$, which is symplectic (see Exercise 56.0.) The next lemma shows that we can complete any basis of $U^\perp \cap W$ into a symplectic basis of $U^\perp$. The union of this basis and the one in U is a symplectic basis with coordinates $(\boldsymbol{q}, \boldsymbol{p})$, in which W can be expressed as advertised. □

Lemma 56.2 *Let U be a null space in a symplectic space V Then one can complete any basis of U into a symplectic basis of V.*

Proof. Without loss of generality, V is $\mathbb{R}2n$ with its standard dot product and canonical symplectic form. Choose an orthornormal basis $(u_1, \ldots, u_l)$ for U. Using the results of Exercise 55.3, the reader can easily check that JU is orthogonal to U (in the sense of the dot product) and that $(u_1, \ldots, u_l, Ju_1, \ldots, Ju_l)$ is a symplectic basis for $E = U \oplus JU$. From Exercise 56.4, $E^\perp \oplus E = V$ and $E^\perp$ is symplectic. We can complete the symplectic basis of E by any symplectic basis of $E^\perp$ and get a symplectic basis for V. □

As a simple consequence of Theorem 56.1, we also get:

Corollary 56.3 *If U is an isotropic subspace of a symplectic space V, one can find a coisotropic W such that $V = U \oplus W$. One can also find a Lagrangian subspace in which U is included.*

This applies in particular to Lagrangian subspaces: given any lagrangian subspace L, we can find another one L' such that $V = L \oplus L'$. In the normal coordinates of the theorem, L would be the $\boldsymbol{q}$ coordinate space, L' the $\boldsymbol{p}$ coordinate space.

Exercise 56.4 Let W be a subspace of a symplectic space V. Show that: W is symplectic $\Longleftrightarrow W \oplus W^\perp = V \Longleftrightarrow W^\perp$ is symplectic (Hint: see the proof of the Linear Darboux theorem).

Exercise 56.5 Show that:
W isotropic $\Longleftrightarrow W \subset W^\perp$.
W coisotropic $\Longleftrightarrow W^\perp \subset W$.
W is Lagrangian $\Longleftrightarrow W$ is a maximal isotropic subspace, or minimal coisotropic subspace (for the inclusion).

Exercise 56.6 This exercise shows how symmetric matrices can be used to locally parametrize Lagrangian planes. Suppose you are given a basis $\boldsymbol{u}_1, \ldots \boldsymbol{u}_n$ for a Lagrangian subspace L of $\mathbb{R}2n$. In the canonical coordinates $(\boldsymbol{q}, \boldsymbol{p})$, write $\boldsymbol{u}_k = (\boldsymbol{u}_k, \boldsymbol{w}_k)$. Let V and W be the $n \times n$ matrices whose columns are the v_k's and w_k's respectively. Suppose that L is a graph over the $\boldsymbol{q}$–plane.

(a) Show that V is invertible and that the column vectors of $\begin{pmatrix} I \\ WV^{-1} \end{pmatrix}$ form a basis for L.
(b) Show that the matrix WV^{-1} is symmetric.
(c) Deduce from this that the (Grassmanian) space of Lagrangian subspaces of $\mathbb{R}2n$ has dimension $n(n+1)/2$.

57. Symplectic Linear Maps

The Linear Darboux Theorem tells us that, up to changes of coordinates, all symplectic vector spaces are identical to $(\mathbb{R}^{2n}, \Omega_0)$. Therefore, as we define and study the transformations that preserve the symplectic form on a vector space, we need only consider the case $(\mathbb{R}^{2n}, \Omega_0)$.

Definition 57.1 A *symplectic linear map* Φ of $(\mathbb{R}^{2n}, \Omega_0)$ is a 1 to 1 linear map which leaves invariant the symplectic form:

$$\Phi^* \Omega_0 = \Omega_0, \quad \text{where } \Phi^* \Omega_0(\boldsymbol{v}, \boldsymbol{w}) := \Omega_0(\Phi \boldsymbol{v}, \Phi \boldsymbol{w}).$$

The group formed by symplectic linear maps is called the *symplectic group* and is denoted by $Sp(2n; \mathbb{R})$, or in short $Sp(2n)$. Because of the Linear Darboux Theorem, this group is naturally identified with the group of $2n \times 2n$ real matrices Φ that satisfy:

$$\Phi^t J \Phi = J, \quad J = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix} \tag{57.1}$$

Examples 57.2

(a) The group Sp(2) is exactly the group of 2×2 matrices of determinant 1.
(b) The transformation $F(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + \boldsymbol{p}, \boldsymbol{p})$, with matrix $\begin{pmatrix} Id & Id \\ 0 & Id \end{pmatrix}$ is symplectic in $\mathbb{R}^{2n}$, and so is any with matrix $\begin{pmatrix} Id & A \\ 0 & Id \end{pmatrix}$, where $A^t = A$. These maps are called *completely integrable* as they preserve the n dimensional foliation of (affine) lagrangian planes $\{\boldsymbol{p} = constant\}$.
(c) A primordial example will be given by the differential of the time 1 map of Hamiltonian flows. (see Section 60.C

Symplectic linear maps have striking spectral properties:

Theorem 57.3 *Symplectic linear maps have determinant 1. If λ is an eigenvalue of a symplectic linear map, so is λ^{-1}, and they appear with the same multiplicity. If λ is a complex eigenvalue, then so are $\lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}$, all with the same multiplicity.*

The origin is a *hyperbolic fixed point* for a linear symplectic map when all the eigenvalues are real and distinct from ± 1. In this case the *stable and unstable manifold* (the n–dimensional union of eigen–subspaces with eigenvalues larger (resp. smaller) than 1 in absolute value) are each n dimensional. These manifolds are also Lagrangian (Exercise 57.6).

Proof. Let Φ be a symplectic map. It is not hard to see that :

$$dq_1 \wedge \ldots \wedge dq_n \wedge dp_1 \wedge \ldots \wedge dp_n = \frac{(-1)^{[n/2]}}{n!} \Omega_0 \wedge \ldots \wedge \Omega_0$$

where $[n/2]$ is the integer part of $n/2$. Since Φ preserves the right hand side of this equation, it must preserves the left hand side, i.e., the volume. Hence $\det\ \Phi = 1$. The rest of the theorem is a consequence of the fact that the characteristic polynomial $C(\lambda)$ of a symplectic transformation Φ has real coefficients and that Φ^t is similar to Φ^{-1}:

$$\Phi^t = J\Phi^{-1}J^{-1}.$$

□

Exercise 57.4 (a) Show that if a $2n \times 2n$ matrix Φ is given by its $n \times n$ block representation:

$$\Phi = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix},$$

then Φ is symplectic if and only if $\boldsymbol{a}\boldsymbol{b}^t = \boldsymbol{b}\boldsymbol{a}^t, \quad \boldsymbol{c}\boldsymbol{d}^t = \boldsymbol{d}\boldsymbol{c}^t, \quad \boldsymbol{a}\boldsymbol{d}^t - \boldsymbol{b}\boldsymbol{c}^t = Id_n$.
(b) Show that

$$\Phi^{-1} = \begin{pmatrix} \boldsymbol{d}^t & -\boldsymbol{b}^t \\ -\boldsymbol{c}^t & \boldsymbol{a}^t \end{pmatrix}.$$

In particular, if Φ is symplectic, so are Φ^{-1} and Φ^t (this can also be shown directly from (57.1) .)

Exercise 57.5 The groups of $2n \times 2n$ real matrices $Gl(n, \mathbb{C})$ and $O(2n)$ are defined by:

$$\Phi \in Gl(n, \mathbb{C}) \Leftrightarrow \Phi J = J\Phi; \quad \Phi \in O(2n) \Leftrightarrow \Phi^t\Phi = Id$$

Show that if Φ is in any two of the groups $Sp(2n), O(2n), Gl(n, \mathbb{C})$, it is in the third. Show that, in this case, we can write:

$$\Phi = \begin{pmatrix} \boldsymbol{a} & -\boldsymbol{b} \\ \boldsymbol{b} & \boldsymbol{a} \end{pmatrix} \quad \text{with} \quad \begin{cases} \boldsymbol{a}^t\boldsymbol{b} = \boldsymbol{b}^t\boldsymbol{a} \\ \boldsymbol{a}^t\boldsymbol{a} + \boldsymbol{b}^t\boldsymbol{b} = Id \end{cases}$$

that is, the complex matrix $\boldsymbol{a} + i\boldsymbol{b}$ is in the unitary group $U(n)$.

Exercise 57.6 (a) Show that, when ± 1 is an eigenvalue of $\Phi \in Sp(n)$, it must appear with even multiplicity.
(b) Show that if λ, λ' are eigenvalues of Φ with eigenvectors $\boldsymbol{v}, \boldsymbol{v}'$ and $\lambda\lambda' \neq 1$ then $\Omega_0(\boldsymbol{v}, \boldsymbol{v}') = 0$.
(c) Deduce from (b) that, if Φ is hyperbolic, its (un)stable manifold is Lagrangian.

Exercise 57.7 Any nonsingular, real matrix Φ has the *polar decomposition*: $\Phi = PO$ where $P = (\Phi\Phi^t)^{1/2}$ is symmetric positive definite, and $O = \Phi P^{-1}$ is orthogonal. (Check this.)
(a) Show that if Φ is symplectic, then P and O are also symplectic.(*Hint.* Prove it for P by decomposing $\mathbb{R}2n$ into eigenspaces for $\Phi\Phi^t$ and using the previous exercise. Notice, in particular, that $O \in U(n)$, by Exercise 57.5.
(b) Show more generally that $(\Phi\Phi^t)^\alpha$ is symplectic for all real α, and deduce from this that $U(n)$ is a deformation retract of $Sp(2n)$.

58. Symplectic Manifolds

Let N be a differentiable manifold. A *symplectic structure* on N is a family of symplectic forms on the tangent spaces of N which depends smoothly on the base point and has a certain nondegeneracy condition. Technically, a symplectic structure is given by a closed nondegenerate differential 2–form Ω:

$$d\Omega = 0 \text{ and, for all } \boldsymbol{v} \neq 0 \in T_zM, \exists \boldsymbol{w} \in T_zM \text{ such that } \Omega(\boldsymbol{v}, \boldsymbol{w}) \neq 0.$$

Ω is called a *symplectic form* and (M, Ω) a *symplectic manifold.* A *symplectic map* or *symplectomorphism* between two symplectic manifolds (N_1, Ω_1) and (N_2, Ω_2) is a differentiable map $F : N_1 \to N_2$ such that:

$$F^*\Omega_2 = \Omega_1.$$

In other words, the tangent space at each point of a symplectic manifold is a symplectic vector space, and the differential of a symplectic map at a point is a symplectic linear map between symplectic vector spaces.

Example 58.1

(a) Once again , the canonical example is given by $(\mathbb{R}^{2n}, \Omega_0)$, where $\mathbb{R}^{2n}$ is thought of as a manifold. The tangent space at a point is identified with $\mathbb{R}^{2n}$ itself, and the form Ω_0 is a constant differential form on this manifold.

(b) Any surface with its volume form is a symplectic manifold. Symplectic maps in dimension 2 are just area preserving maps.

(c) Kähler manifolds (see McDuff & Salamon (1996)) are symplectic.

(d) Cotangent bundles are non compact symplectic manifolds (see Section 59) and time 1 maps of Hamiltonian vector fields on them are symplectic maps.

The fundamental theorem by Darboux (of which we have proven the linear version) says that locally, all symplectic manifolds are isomorphic to $(\mathbb{R}^{2n}, \Omega_0)$. See Arnold (1978), Weinstein (1979) or McDuff & Salamon (1996) for a proof of this.

Theorem 58.2 (Darboux) *Let (N, Ω) be a symplectic manifold. Around each point of N, one can find a coordinate chart $(\boldsymbol{q}, \boldsymbol{p})$ such that :*

$$\Omega = \sum_1^n dq_k \wedge dp_k := d\boldsymbol{q} \wedge d\boldsymbol{p}.$$

Hence all $2n$–dimensional symplectic manifolds are locally symplectomorphic. This is in sharp contrast with Riemannian geometry, where for example the curvature, is an obstruction for two manifolds to be locally isometric.

Submanifolds of a symplectic manifold can inherit the qualitative features of their tangent spaces: A submanifold $Z \subset (N, \Omega)$ is *(co)isotropic* if each of its tangent spaces is (co)isotropic in the symplectic tangent space of N. Hence a *Lagrangian submanifold* is an isotropic submanifold of dimension $n = \frac{1}{2}dimN$. Any curve on a surface is a Lagrangian submanifold. The 0-section and the fiber of the cotangent bundle of a manifold (see next Section) is a Lagrangian submanifold, and so is the graph of any closed differential form.

Exercise 58.3 Show the following:
(a) Any symplectic manifold has even dimension.
(b) If (N, Ω) is a $2n$ dimensional symplectic manifold, then Ω^n is a volume form .
(c) A symplectomorphism is a volume preserving diffeomorphism.

Exercise 58.4 Let (N, Ω) be a symplectic manifold and $F : N \to N$ a symplectomorphism. Show that the set *graph* F is a Lagrangian submanifold of $(N \times N, \Omega \oplus (-\Omega))$

59. Cotangent Bundles

A. Some definitions

Let M be a differentiable manifold of dimension n. Its *cotangent bundle* $T^*M \xrightarrow{\pi} M$ is the fiber bundle whose fiber T_q^*M at a point q of M is the dual to the fiber T_qM of the tangent bundle. The elements of T_q^*M are *cotangent vectors* or linear 1–forms, based at q. Given local coordinates $(q_1, \ldots, q_n)$ in a chart of M , one usually denotes a tangent vector $\boldsymbol{v}$ by:

$$\boldsymbol{v} = \sum_1^n v_k \frac{\partial}{\partial q_k}$$

where $\frac{\partial}{\partial q_k}$ denotes the tangent vector to the k th coordinate line at the point q considered. A cotangent vector p at the point q takes the form:

$$\boldsymbol{p} = \sum_1^n p_k dq_k$$

Where dq_k denotes the 1–form dual to $\frac{\partial}{\partial q_k}$:

$$dq_j(\frac{\partial}{\partial q_k}) = \delta_{jk}.$$

Once the system of coordinates $\boldsymbol{q} = (q_1, \ldots, q_n)$ is chosen, the coordinates $\boldsymbol{p} = (p_1, \ldots, p_n)$ for T_q^*M are uniquely determined, and we call them the *conjugate coordinates*. The cotangent bundle T^*M as a smooth union of the fibers T_q^*M is a differentiable manifold of dimension $2n$, with local coordinates $(\boldsymbol{q}, \boldsymbol{p})$ as presented above. More precisely, if $\boldsymbol{q} \xrightarrow{\Psi} \boldsymbol{Q}$ is a coordinate change between two charts U and V of M, then :

$$(\Psi^*)^{-1}(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P}) = (\Psi(\boldsymbol{q}), (D\Psi_{\boldsymbol{q}}^t)^{-1}\boldsymbol{p})$$

is a change of coordinates in the corresponding charts $U \times \mathbb{R}^n$ and $V \times \mathbb{R}^n$ of T^*M. This law of change of coordinates is what distinguishes tangent vectors from cotangent vectors. More generally, whenever we have a (local) diffeomorphism $F : M \to N$ between two manifolds M and N, there is (locally) an induced *pull–back map*: $F^* : T^*N \to T^*M$ which can be written $F^*(\boldsymbol{q}, \boldsymbol{p}) = (F^{-1}(\boldsymbol{q}), DF_{\boldsymbol{q}}^t(\boldsymbol{p}))$ in coordinates.

Example 59.1
(a)$\mathbb{R}^{2n} \cong \mathbb{R}^n \oplus (\mathbb{R}^n)^*$ can be seen as the cotangent bundle of the manifold $\mathbb{R}^n$: this bundle is trivial, as any bundle over a contractible manifold.
(b)The cotangent bundle of $\mathbb{T}^n$ is $\mathbb{T}^n \times \mathbb{R}^n$. That $T^*\mathbb{T}^n$ is trivial is a consequence of the fact that $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, where $\mathbb{Z}^n$ acts as a group of translations on $\mathbb{R}^n$, whose differentials are the Id. See the following exercise.

Exercise 59.2 More generally, if $M \cong \mathbb{R}^n/\Gamma$ where Γ is a group of diffeomorphisms of $\mathbb{R}^n$ acting *properly discontinuously* (i.e. around each point q of M there is a neighborhood $U(q)$ such that $U \cap (\Gamma \backslash Id)(U) = \emptyset$), then

$$T^*M \cong \mathbb{R}^{2n}/\Gamma^*$$

where Γ^* is the set of diffeomorphisms of $\mathbb{R}^{2n}$ of the form γ^*, where $\gamma \in \Gamma$.

B. Cotangent Bundles as Symplectic Manifold

We now show that there is a natural symplectic structure on T^*M. We first construct a canonical differential 1–form called the *Liouville form* which, we will prove, has the following expression in any set of conjugate coordinates:

$$\lambda = \sum_1^n p_k dq_k = \boldsymbol{p} d\boldsymbol{q}.$$

We then obtain a symplectic form by differentiating λ:

$$\Omega = -d\lambda, \quad \Omega = d\boldsymbol{q} \wedge d\boldsymbol{p},$$

the latter holding in any conjugate coordinate system.

We first present a coordinate free construction of λ. To define a 1–form on T^*M, it suffices to determine how it acts on any given tangent vector v in a fiber $T_\alpha(T^*M)$ of the tangent plane of T^*M. Since the base point α is in T^*M, it is a linear 1–form. Let $\pi : T^*M \to M$ be the canonical projection. The derivative $\pi_* : T(T^*M) \to TM$ takes a vector $\boldsymbol{v}$ to the vector $\pi_*\boldsymbol{v}$ in $T_{\pi(\alpha)}M$. We can evaluate the 1–form α on that vector, and define:

$$\lambda(\boldsymbol{v}) = \alpha(\pi_*\boldsymbol{v})$$

See Figure 59. 5

Fig. 59. 5. The Liouville form on T^*M.

We now compute λ in local, conjugate coordinates. If $(\boldsymbol{q}, \boldsymbol{p})$ are the conjugate coordinates of T^*M, we can write:

$$\alpha = \sum \alpha_k dq_k \text{ and } \boldsymbol{v} = \boldsymbol{u_q}\frac{\partial}{\partial \boldsymbol{q}} + \boldsymbol{u_p}\frac{\partial}{\partial \boldsymbol{p}}.$$

Then $\pi_*(\boldsymbol{v}) = \boldsymbol{u_q}\frac{\partial}{\partial \boldsymbol{q}}$ and $\alpha(\pi_*\boldsymbol{u}) = \sum \alpha_k v_{q_k}$ which exactly says that $\lambda = \boldsymbol{p}d\boldsymbol{q}$. □

The fact that the symplectic form Ω is *exact* ($i.e.$ the differential of another form, here λ) on a cotangent bundle enables us to single out an important class of symplectic map: one way to say that $F : T^*M \to T^*M$ is symplectic in T^*M is to say that the form $F^*\lambda - \lambda$ is closed:

$$d(F^*\lambda - \lambda) = F^*d\lambda - d\lambda = -(F^*\Omega - \Omega) = 0$$

Definition 59.3 A map $F : T^*M \to T^*M$ is *exact symplectic* if $F^*\lambda - \lambda$ is exact:

$$F^*\lambda - \lambda = dS$$

for some real valued function S on T^*M.

We will see in Section 60 that time t maps of flows arising in classical mechanics ($i.e.$ Hamiltonian flows) are all exact symplectic, and so are most of the maps in this book. Note that in $\mathbb{R}2n$, since any closed form is exact, symplectic and exact symplectic are two equivalent properties. On the other hand, the map $(x, y) \to (x, y + a), \quad a \neq 0$, of the cylinder is a good example of a map which is symplectic but not exact symplectic.

Remark 59.4 The term *exact* diffeomorphism, or even *exact symplectic* diffeomorphism is sometimes used to denote the time 1 map of a (time dependent) Hamiltonian system. We will see in Section 60 that, on cotangent bundles, these time–1 maps are indeed exact symplectic in the sense of our definition. It can be shown that the map $(\boldsymbol{q}, \boldsymbol{p}) \mapsto (\boldsymbol{q} + A\boldsymbol{p}, \boldsymbol{p}),\ A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is exact symplectic but not isotopic to Id (true more generally whenever A is not homotopic (cannot be deformed) to I on $\mathbb{T}^2$). Hence these maps cannot be time–1 maps of Hamiltonians. Cotangent bundles are just one example, albeit the most important one, of *exact symplectic manifolds*: symplectic manifolds whose symplectic form is exact. Many facts that are true for cotangent bundles also hold for exact symplectic manifolds.

Exercise 59.5 Show that the set of exact symplectic maps forms a group under composition. In particular, show that generating functions if $G * \lambda - \lambda = S_G$ and $F^*\lambda - \lambda = S_F$ then

$$(F \circ G)^*\lambda - \lambda = d[(S_F \circ G) + S_G]$$

Exercise 59.6 Let $F : T^*\mathbb{S}^1 \to T^*\mathbb{S}^1$. The *net flux* of F through a non contractible simple closed loop C is the difference between the area above C but below $F(C)$, and the area above $F(C)$ but below C.
(a) Show that, if F is symplectic, the net flux is independent of the choice of C.
(b) Show that if F is symplectic then: F is exact symplectic $\Leftrightarrow$ F has zero net flux . In particular, an area preserving map of the annulus that has an invariant circle is automatically exact symplectic.

Exercise 59.7 Show that a map F of T^*M is exact symplectic if and only if :

$$\int_{F\gamma} \boldsymbol{p}d\boldsymbol{q} = \int_{\gamma} \boldsymbol{p}d\boldsymbol{q}$$

for all differentiable closed curve γ.

C. Notable Lagrangian Submanifolds of Cotangent Bundles

It is not hard to see that the fibers of T^*M are Lagrangian submanifolds of T^*M: in coordinates they are given by $\{\boldsymbol{q} = \boldsymbol{q}_0\}$ and hence their tangent space is of the form $\{\boldsymbol{q} = 0\}$. Likewise, the zero section 0^*_M of T^*M is Lagrangian. Another class of example will be of importance to us in Chapter CZ. Consider a function $W : M \to \mathbb{R}$. Its differential dW can be seen as a section of T^*M, i.e. a map $M \to T^*M$ whose image $dW(M)$ can be written as $\{(\boldsymbol{q}, dW(\boldsymbol{q})) \mid \boldsymbol{q} \in M\}$. A basis for the tangent space of $dW(M)$ at a point $(\boldsymbol{q}, dW(\boldsymbol{q}))$ is given by:

$$\boldsymbol{v}_k = \frac{\partial}{\partial q_k} + \sum_{j=1}^{n} \frac{\partial^2 W(\boldsymbol{q})}{\partial q_j \partial q_k} \frac{\partial}{\partial p}_j$$

It is not hard to see that:

$$\Omega(\boldsymbol{v}_k, \boldsymbol{v}_l) = \frac{\partial^2 W(\boldsymbol{q})}{\partial q_k \partial q_l} - \frac{\partial^2 W(\boldsymbol{q})}{\partial q_l \partial q_k} = 0,$$

so that $dW(M)$ *is a Lagrangian submanifold of* M. We can generalize this argument somewhat. Any 1–form α can be seen as a map from M to T^*M, so we can ask the question: for what α is $\alpha(M)$ a Lagrangian manifold ? To answer this question, one can check (Exercise 59.8) the following formula:

(59.1) $$\alpha^*\lambda = \alpha.$$

where λ is the Liouville form (the reader has to get used to the fact that we see α either as a form or a map, at our convenience. When seen as a map, α is actually an embedding of M into T^*M.)

The manifold $\alpha(M)$ is Lagrangian exactly when:

$$0 = \alpha^*\Omega = \alpha^*(-d\lambda) = -d(\alpha^*\lambda) = -d\alpha,$$

that is, exactly when α is a closed form. In particular, if the form α is exact with $\alpha = dW$, this gives another proof that $dW(M)$ is Lagrangian. W is the simplest instance of *generating function* for the Lagrangian manifold $\alpha(M) = dW(M)$ (*generating phase* or *generating phase function* is also used). We will expend on this important notion of symplectic topology in Chapter CZ.

Exercise 59.8 Verify Formula (59.1) , using local coordinates.

60. Hamiltonian Systems

A. Lagrangian Systems versus Hamiltonian systems

A lot of mechanical problems can be put in terms of a variational problem: under the *principle of least action*, trajectories are critical points of an *action functional* of the form:

$$A(\gamma) = \int_{t_0}^{t_1} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)dt,$$

with boundary condition $\gamma(t_0) = \boldsymbol{q}_0, \gamma(t_1) = \boldsymbol{q}_1$. The function L is twice differentiable in each variables, say (absolute continuity is enough). It is called the *Lagrangian function* of the system. As this is a somewhat heuristic discussion, we will not specify here the functional space to which γ belongs. In concrete cases (say $\gamma \in C^1([t_0, t_1]), \mathbb{R}^n)$ or $C^1([t_0, t_1], M)$, or some Sobolev space of curves...), the following can be made quite rigorous.

To compute the differential of A, one applies a small variation $\delta\gamma = (\delta\boldsymbol{q}, \delta\boldsymbol{p})$ to γ, with $\delta\gamma(t_0) = \delta\gamma(t_1) = 0$. Then:

$$\delta A(\gamma) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \boldsymbol{q}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\delta\boldsymbol{q} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\delta\dot{\boldsymbol{q}} \right) dt.$$

performing an integration by parts on the second term of this integral, we get:

$$\delta A(\gamma) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} \right) \delta\boldsymbol{q}dt$$

Since this should be true for any variation $\delta\gamma$, we must have:

(60.1) $$\frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} = 0,$$

which is a second order differential equation in $\boldsymbol{q}$ called the *Euler- Lagrange* equations. (The plural to "equations" just refers to the fact that the dimension is usually greater than 1.) As an example, a large number of mechanical systems have a Lagrangian function of the form:

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \frac{1}{2}\|\dot{\boldsymbol{q}}\|^2 - V_t(\boldsymbol{q}).$$

("Kinetic - potential". The time dependance of V usually refers to some forcing) where $V : \mathbb{R}^n \to \mathbb{R}, f : \mathbb{R} \to \mathbb{R}^n$. The Euler–Lagrange equations for such a system are:

$$\ddot{\boldsymbol{q}} + \nabla V_t(\boldsymbol{q}) = 0$$

To solve such an O.D.E., one usually proceeds by introducing $\boldsymbol{p} = \dot{\boldsymbol{q}}$ to get a system of first order ODE's:

$$\dot{\boldsymbol{q}} = \boldsymbol{p}$$
$$\dot{\boldsymbol{p}} = -\nabla V(\boldsymbol{q}).$$

As we will see presently, we have just put the Lagrangian problem into a Hamiltonian form. In general, if

$$\det \frac{\partial^2 L}{\partial \dot{\boldsymbol{q}}^2} \neq 0, \tag{60.2}$$

we can introduce

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}$$

to transform the Euler–Lagrange equations (60.1) into a system of first order O.D.E.'s: because of the nondegeneracy condition (60.2) , the implicit function theorem implies that, locally, we can make a change of variables :

$$\mathcal{L} : (\boldsymbol{q}, \dot{\boldsymbol{q}}) \to (\boldsymbol{q}, \boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}) \tag{60.3}$$

This is, when $\boldsymbol{q}$ is seen as a point on a manifold M, a local diffeomorphism between $T_{\boldsymbol{q}}M$ and $T^*_{\boldsymbol{q}}M$. This change of variables is called the *Legendre transformation.*[16]

Define the *Hamiltonian function* by:

$$H(\boldsymbol{q}, \boldsymbol{p}, t) = \boldsymbol{p}\dot{\boldsymbol{q}} - L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t),$$

Where it is understood that $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}} \circ \mathcal{L}^{-1}(\boldsymbol{q}, \boldsymbol{p})$. We can compute:

$$\begin{aligned} \frac{\partial H}{\partial \boldsymbol{q}} &= \boldsymbol{p}\frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{q}} - \frac{\partial L}{\partial \boldsymbol{q}} - \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{q}} = -\frac{\partial L}{\partial \boldsymbol{q}}, \\ \frac{\partial H}{\partial \boldsymbol{p}} &= \dot{\boldsymbol{q}} + \boldsymbol{p}\frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{p}} - \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{p}} = \dot{\boldsymbol{q}}. \end{aligned}$$

But the Euler-Lagrange equations imply that:

$$\dot{\boldsymbol{p}} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \frac{\partial L}{\partial \boldsymbol{q}} = -\frac{\partial H}{\partial \boldsymbol{q}}.$$

Combining this with the previous formula yields *Hamilton's equations*:

$$\begin{aligned} \dot{\boldsymbol{q}} &= H_{\boldsymbol{p}} \\ \dot{\boldsymbol{p}} &= -H_{\boldsymbol{q}}. \end{aligned} \tag{60.4}$$

Remark 60.1

(a) The Legendre transformation is *involutive*: it is its own inverse, in the following sense. The map

[16] In the classical literature the term Legendre transformation refers to the complete process of changing the Lagrangian L into the Hamiltonian H as shown in this section, and H is then called the Legendre transformation of L. It is grammatically less awkward to call H the Legendre *transformed* of L.

$(\boldsymbol{q}, \dot{\boldsymbol{q}}) \to (\boldsymbol{q}, \frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{p})$ has inverse:

$$(\boldsymbol{q}, \boldsymbol{p}) \to (\boldsymbol{q}, \frac{\partial H}{\partial \boldsymbol{p}} = \dot{\boldsymbol{q}})$$

and L is the Legendre transformed of H in the sense that:

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \boldsymbol{p}\dot{\boldsymbol{q}} - H(\boldsymbol{q}, \boldsymbol{p}, t)$$

where $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ is given implicitly by $\frac{\partial H}{\partial \boldsymbol{p}} = \dot{\boldsymbol{q}}$.
(b) In the new coordinates, the action functional becomes:

$$A(\gamma) = \int_\gamma \boldsymbol{p} d\boldsymbol{q} - H dt$$

where γ is seen as a curve $(\boldsymbol{q}(t), \boldsymbol{p}(t), t)$ in the space $\mathbb{R}2n \times \mathbb{R}$, or $T^*M \times \mathbb{R}$.

Hamilton's equations have a natural expression in the symplectic setting. We assume now that $\boldsymbol{q}$ is in $\mathbb{R}^n$. Using the notation $H_t(\boldsymbol{q}, \boldsymbol{p}) = H(\boldsymbol{q}, \boldsymbol{p}, t)$, we can rewrite (60.4) as

$$\dot{z} = -J\nabla H_t(z) \overset{def}{:=} X_H(z, t).$$

where $\nabla H_t = \binom{H_q}{H_p}$ is the gradient of H_t with respect to the scalar product on $\mathbb{R}^{2n}$:

$$\langle \nabla H_t, \boldsymbol{v} \rangle = dH_t(\boldsymbol{v})$$

Likewise, X_H, which we call the *Hamiltonian vector field* should be seen as the *symplectic gradient* of H_t:

$$\Omega_0(X_H, \boldsymbol{v}) = \langle -J^2 \nabla H_t, \boldsymbol{v} \rangle = \langle \nabla H_t, \boldsymbol{v} \rangle = dH_t(\boldsymbol{v}).$$

This can be written using the contraction operator on differential forms:

$$i_{X_H} \Omega = dH_t$$

Exercise 60.2
(a) Compute the Legendre transformed of $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \frac{1}{2}\langle A\dot{\boldsymbol{q}}, \dot{\boldsymbol{q}} \rangle - V(\boldsymbol{q})$.
(b) Show that, in general, if H is the Legendre transformed of L, then

$$L_{\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}} H_{\boldsymbol{pp}} = Id.$$

B. Hamiltonian Systems on a Symplectic Manifold

Motivated by the last expression that we found for the Hamiltonian vector field in $\mathbb{R}^{2n}$, we extend the definition to symplectic manifolds:

Definition 60.3 Let (N, Ω) be a symplectic manifold and $H(z, t) = H_t(z)$ be a real valued function on $N \times \mathbb{R}$. The *Hamiltonian vector field* associated with H is the (time dependent) vector field X_H defined by:

$$\Omega(X_H, \boldsymbol{v}) = dH_t(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in TM.$$

Equivalently:

$$i_{X_H} \Omega = dH_t$$

The (time dependent) O.D.E.:

$$\dot{\boldsymbol{z}} = X_H(\boldsymbol{z}, t) \tag{60.5}$$

is called *Hamilton's equations.*

In local Darboux coordinate charts (*eg.* in conjugate coordinates chart of a cotangent bundles), these equations take the form of (60.4) . If H is time independent, then (60.5) generates a (local) flow on N.If H is time dependent, then X_H generates a (local) flow in the space $N \times \mathbb{R}$, called the *extended phase space* in mechanics. Specifically, one solves the following time independent system on $N \times \mathbb{R}$:

$$\dot{\boldsymbol{z}} = X_H(\boldsymbol{z}, s)$$
$$\dot{s} = 1$$

which generates a flow ϕ^t in $N \times \mathbb{R}$ satisfying:

$$\phi^t(\boldsymbol{z}, s) = (h_s^{t+s}(\boldsymbol{z}), s + t),$$

where h_s^t is a family of C^{k-1} diffeomorphims of N ,depending C^{k-1} on s and t. This is a general procedure for time dependent vector field. The diffeomorphism h_s^t is called a *Hamiltonian map* and, for each fixed s the curve $t \to h_s^t$ is a *Hamiltonian isotopy* (an *isotopy* is a smoothly varying 1–parameter family of diffeomorphisms). Another way of describing $h_s^t(\boldsymbol{z})$ is by saying that it is the unique solution $\boldsymbol{z}(t)$ of Hamilton's equation with initial condition $\boldsymbol{z}(s) = \boldsymbol{z}$. In practice, one often fixes $s = 0$ and denotes h_0^t by h^t.

The following exercise shows the one to one correspondence between time dependent vector fields and isotopies. It also shows that, even though the time 0 of a solution flow to a time dependent vector field is the Identity, the flow does not in general form a group.

Exercise 60.4 Let X_t be a vector field (not necessarily Hamiltonian) on a manifold N. Let h_s^t be the solution flow to the O.D.E. $\dot{\boldsymbol{z}} = X_t(\boldsymbol{z}), \dot{s} = 1$. Prove that:
(i) $h_s^s = Id, \ \forall s$,
(ii)$h_s^{t'} = h_t^{t'} \circ h_s^t$, so that in particular $h_s^t = h^t \circ (h^s)^{-1}$. Compute $(h_s^t)^{-1}$.
(iii) Conversely, given any (sufficiently smooth) isotopy g^t in N, with $g^0 = Id$, show that the time dependent vector field:

$$\dot{g}^t = \frac{d}{ds}\Big|_{s=0} g^{t+s} \circ (g^t)^{-1}$$

has solution $h_s^t = g^t \circ (g^s)^{-1}$.

C. Invariants of the Hamiltonian Flow

We analyze here how different objects vary under the Hamiltonian flow. If G is a function on a differentiable manifold N, and X is a vector field, we recall that the *Lie derivative* of G along X is:

$$L_X G(z) = \frac{d}{dt}\Big|_{t=0} G(\phi^t(z)) = dG(X(z))$$

where ϕ^t is the flow solution for X .

Theorem 60.5 *Let H be a time independent Hamiltonian function on (M, Ω) . Then H is constant under the Hamiltonian flow it generates:*

$$L_{X_H} H = 0.$$

Proof. $L_{X_H} H = dH(X_H) = \Omega(X_H, X_H) = 0$ □

Remark 60.6 $L_{X_H} G = \Omega(X_G, X_H) = -L_{X_G} H$ is also denoted by $\{G, H\}$ and it is called the *Poisson bracket* of H and G. Hence, the poisson bracket measures how far the function G (resp. H) is from being constant along the flow of X_H (resp.X_G). When $\{H, G\} = 0$, one says that G (resp. H) is a *first integral* of the Hamiltonian flow of H (resp. G), or that the functions H and G are *in involution.* One can show (see *eg.* Arnold (1978), Abraham & Marsden (1985)) that the set of Hamiltonian vector fields form a Lie sub– algebra of the Lie algebra of vector fields on a manifold, in the sense that:

$$X_{\{H,G\}} = [X_H, X_G].$$

In particular, the poisson bracket of two functions measures how far from commuting their Hamiltonian flows are.

One can also compute how a differential form α varies along an isotopy g_t by the Lie derivative. Let us first extend the notion of Lie derivative to differential forms. If X is any vector field, we define *the Lie derivative in the direction of X* by:

$$L_X \alpha = \left.\frac{d}{dt}\right|_{t=0} g_t^* \alpha.$$

where g_t is the flow generated by X. At time $t \neq 0$,

$$\frac{d}{dt} g_t^* \alpha = g_t^* L_X \alpha.$$

Hence, the isotopy g_t preserves the form α whenever this Lie derivative is zero:

$$g_t^* \alpha = \alpha, \ \forall t \Longleftrightarrow L_X \alpha = 0.$$

We have the important *homotopy formula* (see *eg.* McDuff & Salamon (1996)):

$$L_X \alpha = i_X d\alpha + d(i_X \alpha) \tag{60.6}$$

and again, at time $t \neq 0$,

$$\frac{d}{dt} g_t^* \alpha = g_t^* \left(i_X d\alpha + d(i_X \alpha) \right)$$

A *symplectic isotopy* g_t on (M, Ω) is an isotopy such that g_t is a symplectic map for all t. By the homotopy formula (and the fact that a symplectic form is closed), this can be reworded:

$$g_t \text{ is a symplectic isotopy} \Longleftrightarrow L_X \Omega = 0 \Longleftrightarrow d(i_X \Omega) = 0 \tag{60.7}$$

The following theorem characterises Hamiltonian isotopies, at least in cotangent bundles (or in any exact symplectic manifold, i.e. one whose symplectic form is exact)

Theorem 60.7 *(a) On any symplectic manifold, Hamiltonian isotopies are symplectic.*
(b) On a cotangent bundle T^*M, a Hamiltonian isotopy with Hamiltonian $H(\boldsymbol{z},t)$ is also exact symplectic:

$$h^{t^*}\lambda - \lambda = h^{t^*}\boldsymbol{p}d\boldsymbol{q} - \boldsymbol{p}d\boldsymbol{q} = dS_t, \quad \text{with } S_t = \int_\gamma \boldsymbol{p}d\boldsymbol{q} - Hd\tau$$

where γ is the curve $(h^\tau(\boldsymbol{z}),\tau)$, $\tau \in [0,t]$ solution of Hamilton's equations in the extended phase space $T^*M \times \mathbb{R}$, and $\boldsymbol{z}$ is the point at which the form is evaluated.
(c) Conversely, if an isotopy g_t is exact symplectic then it is Hamiltonian, with the Hamiltonian function given by:

$$H_t = i_{X_t}\boldsymbol{p}d\boldsymbol{q} - (g_t^{-1})^*\frac{d}{dt}(S_t).$$

where $X_t(\boldsymbol{z}) = \frac{dg_t}{dt}(g_t^{-1}(\boldsymbol{z}))$.

Proof. The first assertion (a) is an immediate consequence of (60.7) : if h^t is a Hamiltonian isotopy then $i(\dot{h}_t)\Omega = dH_t$ is exact, and therefore closed. In cotangent bundles, it is also a consequence of the second assertion. We look for $\frac{d}{dt}(S_t)$ in the statement (b):

$$\frac{d}{dt}h_t^*\lambda = h_t^*\left(i_{X_H}d\lambda + d(i_{X_H}\lambda)\right) = h_t^*d(-H_t + i_{X_H}\lambda) = dh_t^*(-H_t + i_{X_H}\lambda)$$

From this we get:

$$h_t^*\lambda - \lambda = d\int_0^t h_\tau^*(-H_\tau + i_{X_H}\lambda)d\tau \stackrel{\text{def}}{=} dS_t$$

that is, h^t is exact symplectic. We leave it to the reader to rewrite the integral as the one advertised in the theorem. This finishes the proof of (b). To prove the converse (c), let g_t be an exact symplectic isotopy:

$$g_t^*\boldsymbol{p}d\boldsymbol{q} - \boldsymbol{p}d\boldsymbol{q} = dS_t$$

for some S_t differentiable in all of $(\boldsymbol{q},\boldsymbol{p},t)$. We claim that the (time dependent) vector field:

$$X_t(\boldsymbol{z}) = \frac{dg_t}{dt}(g_t^{-1}(\boldsymbol{z}))$$

whose time t is g_t, is Hamiltonian. To see this, we compute:

$$\frac{d}{dt}(dS_t) = \frac{d}{dt}g_t^*\boldsymbol{p}d\boldsymbol{q} = g_t^*L_{X_t}\boldsymbol{p}d\boldsymbol{q} = g_t^*\left(i_{X_t}d(\boldsymbol{p}d\boldsymbol{q}) + d(i_{X_t}\boldsymbol{p}d\boldsymbol{q})\right),$$

from which we get

$$i_{X_t}d\boldsymbol{q}\wedge d\boldsymbol{p} = d\left(i_{X_t}\boldsymbol{p}d\boldsymbol{q} - (g_t^{-1})^*\frac{d}{dt}(S_t)\right) = dH_t$$

which exactly means that X_t is Hamiltonian with H_t as Hamiltonian function. □

A less formal proof of (b) in the above theorem yields extra information. We follow Chapter 9 in Arnold (1978). We first prove that the vector field $(X_H, 1)$ in $T^*M \times \mathbb{R}$ generates the kernel of the form $d(\boldsymbol{p}d\boldsymbol{q} - Hdt) = d\boldsymbol{p}\wedge d\boldsymbol{q} - H_{\boldsymbol{q}}d\boldsymbol{q}\wedge dt - H_{\boldsymbol{p}}d\boldsymbol{p}\wedge dt$. The matrix of this form in the (Darboux) coordinate $(\boldsymbol{q},\boldsymbol{p},t)$ is:

$$A = \begin{pmatrix} 0 & -Id & H_{\boldsymbol{q}} \\ Id & 0 & H_{\boldsymbol{p}} \\ -H_{\boldsymbol{q}} & -H_{\boldsymbol{p}} & 0 \end{pmatrix}.$$

since the upper left $2n \times 2n$ matrix is the nonsingular matrix J, A is of rank (at least) $2n$. It is easy to see that the vector $(H_{\boldsymbol{p}}, -H_{\boldsymbol{q}}, 1)$ generates its kernel.

Now, take a closed curve γ in $T^*M \times \mathbb{R}$. The image under the Hamiltonian flow of γ forms an embedded tube in $T^*M \times \mathbb{R}$. Since the tangent space to this tube at any of its point $\boldsymbol{z}$ contains the vector $X_H(\boldsymbol{z})$, the form $d(\boldsymbol{p}d\boldsymbol{q} - Hdt)$ restricted to this tube is null. As a result, because of Stokes' theorem, if γ_1 and γ_2 in $T^*M \times \mathbb{R}$ encircle the same tube of orbits of the extended flow, we must have:

$$\int_{\gamma_1} \boldsymbol{p}d\boldsymbol{q} - Hdt = \int_{\gamma_2} \boldsymbol{p}d\boldsymbol{q} - Hdt \tag{60.8}$$

since $\gamma_1 - \gamma_2$ is the boundary of a region of the tube. The form $\boldsymbol{p}d\boldsymbol{q} - Hdt$ is called the *integral invariant of Poincaré–Cartan*. As a particular case, if γ_1 is of the form (γ, t_1) and $\gamma_2 = (h_{t_1}^{t_2}\gamma, t_2)$, the form Hdt is null on these curves and hence Equation (60.8) reads:

$$\int_{\gamma_1} \boldsymbol{p}d\boldsymbol{q} = \int_{h_{t_1}^{t_2}\gamma} \boldsymbol{p}d\boldsymbol{q} \tag{60.9}$$

This last equation implies the statement (b) in Theorem 60.7: it proves that the function

$$S_t = \int_{\boldsymbol{z}_0}^{\boldsymbol{z}} h^{t*}\boldsymbol{p}d\boldsymbol{q} - \boldsymbol{p}d\boldsymbol{q} \tag{60.10}$$

is well defined, i.e. the integral does not depend on the path chosen between z_0 and z. This proves in turn that h^t is exact symplectic. □

The next theorem, due to Jacobi, runs somewhat against the title of this subsection, in the sense that we show that symplectic diffeomorphisms conserve Hamilton's equations. This property in fact characterises symplectic transformations, which are for this reason called *canonical transformations* in the classical litterature. Even though we will not need this theorem in the sequel, we include it here since it explains why symplectic geometry came to exist.

Theorem 60.8 *Let $F : (M, \omega_M) \to (N, \omega_N)$ be a diffeomorphism. Then F is symplectic if and only if for all function $H : N \to \mathbb{R}$,*

$$F_* X_{H \circ F} = X_H. \tag{60.11}$$

In this case, F conjugates the Hamiltonian flows h^t and g^t of H and $H \circ F$ respectively:

$$g^t = F^{-1} \circ h^t \circ F.$$

This holds also when H is time dependent.

Proof. Reminding the reader that by definition $F_*X(F(z)) = DF_z X(z)$ for any vector field X, we also use the notation F^*Y to mean $(F^{-1})_*Y$. It is not hard to check that the following formula holds:

$$F^* i_X \alpha = i_{F^*X} F^* \alpha \tag{60.12}$$

for any vector field X and differential form α. Coming back to our statement, we have on one hand:

$$F^* i_{X_H} \omega_N = F^* dH = dH \circ F$$

by tracking down definitions, and on the other hand,

$$F^* i_{X_H}\omega_N = i_{F^*X_H}F^*\omega_N = i_{F^*X_H}\omega_M$$

because of (60.12) and the fact that F is symplectic. This proves (60.11) . Conversally, if (60.11) holds for any H, the same kind of computation shows that,

$$i_{X_{H\circ F}}F^*\omega_N = i_{X_{H\circ F}}\omega_M$$

and since any tangent vector at a point of M is of the form $X_{H\circ F}$ for some H, we must have $F^*\omega_N = \omega_M$, i.e. F is symplectic. The conjugacy statement, a general fact about O.D.E.'s, is left to the reader, as well as checking that everything still works with time dependent systems. □

Exercise 60.9 The Lie derivative of a function can be defined, in the obvious way, along any differentiable isotopy. What fails in Theorem 60.5 when H is time dependent?

Exercise 60.10 Show that in Darboux coordinates:

$$\{H, G\} = \frac{\partial H}{\partial \boldsymbol{q}}\frac{\partial G}{\partial \boldsymbol{p}} - \frac{\partial H}{\partial \boldsymbol{p}}\frac{\partial G}{\partial \boldsymbol{q}}.$$

Exercise 60.11 Prove that the function S_t defined in (60.10) satisfies:

$$S_t(\boldsymbol{z}) = \int_\gamma \boldsymbol{p}d\boldsymbol{q} - Hdt \quad + C(\boldsymbol{z}_0, t),$$

for some C, and γ as in Theorem 60.10. (*Hint.* Apply Stokes on the appropriate surface.)

Exercise 60.12 Prove that h_s^t is exact symplectic (i.e. even for $s \neq 0$), where $h_s^t(\boldsymbol{z})$ is, as in subsection B, the solution of Hamilton's equation such that $\boldsymbol{z}(s) = \boldsymbol{z}$.

Exercise 60.13 Let H be autonomous , or of period τ. Show that $X_H(\boldsymbol{z})$ is preserved by $Dh^\tau(\boldsymbol{z})$, *i.e.* X_H is an eigenvector of Dh^τ with eigenvalue 1.

D. Symplectic Maps as Return Maps of Hamiltonian Systems

Consider a time independent Hamiltonian on $\mathbb{R}^{2n+2}$, with its standard symplectic structure $\Omega_0 = \sum_{k=0}^n dq_k \wedge dp_k$. Assume that we have a periodic trajectory γ for the Hamiltonian flow. It must then lie in an energy level $H = H_0 = H(\gamma(0))$, since H is time independent. Take any $2n+1$ dimensional open disk $\tilde{\Sigma}$ which is transverse to γ at $\gamma(0)$, and such that $\tilde{\Sigma}$ intersects γ only at $\gamma(0)$.

Such a disk clearly always exists, if γ is not a fixed point. In fact, one can assume that, in a local Darboux chart, $\tilde{\Sigma}$ is the hyperplane with equation $q_0 = 0$: this is because in the construction of Darboux coordinates, one can start by choosing an arbitrary nonsingular differentiable function as one of the coordinate function (see [Ar78], section 43, or [We77], Extension Theorem, lecture 5.)

Define $\Sigma = \tilde{\Sigma} \cap \{H = H_0\}$. It is a standard fact (true for periodic orbits of general flows) that the Hamiltonian flow h^t admits a Poincaré return map $\mathcal{R}$, defined on Σ around $\boldsymbol{z}_0$, by $\mathcal{R}(\boldsymbol{z}) = h^{t(\boldsymbol{z})}(\boldsymbol{z})$, where $t(\boldsymbol{z})$ is the first return time of $\boldsymbol{z}$ to Σ under the flow (see Hirsh & Smale (1974), Chapter 13).

We claim that $\mathcal{R}$ is symplectic, with the symplectic structure induced by Ω_0 on Σ.

Since $\tilde{\Sigma}$ is transverse to γ, we may assume that:

$$\dot{q}_0 = \frac{\partial H}{\partial p_0} \neq 0$$

Fig. 60. 2.

on $\tilde{\Sigma}$. Hence, by the Implicit Function Theorem, the equation

$$H(0, q_1 \ldots, q_n, p_0, \ldots, p_n) = H_0$$

implies that p_0 is a function of $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. This makes the latter variables a system of local coordinates for Σ, and since $dq_0 = 0$ on Σ, the restriction of Ω_0 is in fact

$$\omega = \Omega_0\big|_{\Sigma} = \sum_{k=1}^{n} dq_k \wedge dp_k.$$

To prove that $\mathcal{R}$ is symplectic, remember that, by (60.9) , for any closed curve in Σ, or more generally for any closed 1–chain c in Σ,

$$\int_{\mathcal{R}c} \boldsymbol{p}d\boldsymbol{q} - H dt = \int_{c} \boldsymbol{p}d\boldsymbol{q} - H dt$$

since c and $\mathcal{R}c$ are on the same trajectory tube. Here $\mathcal{R}c$ represent the chain in $\mathbb{R}^{2n+2} \times \mathbb{R}$ given by $(\mathcal{R}(c(s)), t^{c(s)})$.This equality implies that the function $S(\boldsymbol{z}) = \int_{\boldsymbol{z}_0}^{\boldsymbol{z}} \mathcal{R}^*(\boldsymbol{p}d\boldsymbol{q} - H dt) - (\boldsymbol{p}d\boldsymbol{q} - H dt)$ is well defined. But, on Σ, the differential of the form inside this integral is $\mathcal{R}^*\omega - \omega$, since both dq_0 and dH are zero there. Hence $\mathcal{R}^*\omega - \omega = d^2S = 0$, i.e., $\mathcal{R}$ is symplectic.

Remark SGleginv is 60.1,Theorem SGthmcanva is 56.1Theorem SGhamexactsymp is 60.7, Formula SGintinv is (60.9) , Formula SGst is (60.10) , Exercise SGexoexactsympcurve is 59.7, Exercise SGexoxhev is 60.13, Exercise SGexoisotopy is 60.4, Formula SGformhomotopy is (60.6) , Exo SGexolagsym is 56.6.