Thickness-2 Complexes are 3-Colorable

The Coloring Clique:
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April 4, 2011

Abstract
First draft of proof. Based on our meeting 31Mar11.

1 Definition
A complex is of thickness-\(k\) if it consists of bricks each of which has a \(z\)-height exactly 1, and the total \(z\)-thickness of the complex is \(k\). The \(x\)- and \(y\)-extents of the bricks are arbitrary integers. We assume a complex has a connected dual graph.

2 Theorem & Proof
Theorem 1 Every thickness-2 complex is 3-colorable.

Proof: A thickness-2 complex \(C\) can have portions of thickness-1 on the bottom layer, or on the top layer, as well as holes straight through. See Figure 1. The plan of the proof is to fill all thickness-1 sections (on either layer) to become thickness-2, and then to apply the Duplicate Layer reduction.

If \(C\) has no thickness-1 sections, then it must be that all portions are thickness-2 or holes through both layers. Therefore the Duplicate Layer reduction applies, removing the top layer. Then we are left with a thickness-1 complex, which is planar and therefore 3-colorable (or we could reduce it to the empty complex). Either way, we obtain a 3-coloring of \(C\).

So assume \(C\) has some thickness-1 sections. Without loss of generality, assume there is a thickness-1 section on the bottom layer. Let \(b_0\) be a brick in such a section on the bottom layer. It must be adjacent to one or more bricks on the bottom layer (because \(C\) is connected). We consider two cases.

Case 1. None of the bricks on the bottom layer to which \(b_0\) is adjacent have a brick on top of them. Then add a brick \(b'_0\) of \(z\)-height 1 on top of \(b_0\). The result is a new complex \(C_1\).
Case 2. $b_0$ is adjacent to at least one brick $b_1$ on the bottom which has a brick $b'_1$ on top of it. See Figure 1. We now argue that adding a brick $b'_0$ of $z$-height 1 on top of $b_0$ will glue properly to $b'_1$. The same argument holds for other bricks with the same adjacency properties as $b_1$ and $b'_1$.

We know $b_0$ and $b_1$ share a face on the lower level. This means they have the same extent in either $x$ or $y$; say $x$ without loss of generality, as is illustrated in Figure 1. Because $b'_0$ is on top of $b_0$, it must have the same $xy$-dimensions as $b_0$. Then, because $b'_1$ has the same $xy$-dimensions as $b_1$ (because it is on top of $b_1$), and $b_1$ has the same $x$-extent as $b_0$, $b'_0$ matches $b'_1$ on its $xz$-face, and so constitutes a proper gluing, creating a new complex $C_1$.

Continuing in this manner, we can “fill” all thickness-1 sections, in either layer, until no more fills are available. Let the resulting complex be $C'$. $C'$ consists only of thickness-2 sections and holes right through. Therefore its top and bottom layers are identical, and the reduction described above applies and shows $C'$ is 3-colorable. Finally, because $C$ is inside $C'$, i.e., $C \subseteq C'$, $C$ itself is 3-colorable.

\[ \square \]