Minimum-Perimeter Enclosing \( k \)-gon

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**Introduction** Let \( P = p_1, \ldots, p_n \) be a simple polygon (all polygons are assumed convex throughout this paper). A fundamental problem in geometric optimization is to compute a minimum-area or a minimum-perimeter convex \( k \)-gon (denoted \( Q^A \) or \( Q^p \), resp.) that encloses \( P \). While efficient algorithms for finding \( Q^A \) are known for more than 20 years \([8,12]\), the problem of finding \( Q^p \) has remained open; the problem is posed as open in \([3,6,7,9,12,14,5]\). Chang and Yap \([8]\) give a comprehensive classification of the inclusion/enclosure problems, but do not mention the minimum-perimeter enclosing \( k \)-gon problem \((\text{Enc}(P_{\text{all}}, P_k, \text{perimeter}), \text{in their terminology}) \) at all.

We give the first polynomial-time algorithms for computing \( Q^p \). In order to obtain our solution, we prove a structural result about an optimal polygon: Local optimality implies that it is “flush” with \( P \) (Lemma 1). As a by-product we obtain an algorithm for finding the minimum-perimeter “envelope” — a convex \( k \)-gon with a specified sequence of interior angles. Our proofs are very simple and are based on elementary geometry\(^2\).

The exact coordinates of the vertices of \( Q^p \) are given by the roots of high-degree polynomials. In general, it is impossible to find the coordinates exactly in polynomial time \([4]\). Thus, given \( \varepsilon > 0 \) as a part of the input to the problem, we will be satisfied with a \((1 + \varepsilon)\)-approximate solution.

**Finding \( Q^p \)** Our algorithm is based on the following lemma, whose (simple) proof we defer until the next paragraph:

**Lemma 1.** \( Q^p \) is flush with \( P \), i.e., one of the edges of \( Q^p \) contains an edge of \( P \).

By Lemma 1 we may consider each edge of \( P \) as a candidate flush edge with \( Q^p \) and turn the scene into simple polygon \( \overline{P} \) (Fig. 1). This reduces finding \( Q^p \) to solving \( n \) instances of the problem of finding a shortest \((k + 1)\)-link path in simple polygon \( \overline{P} \), a problem which can be solved in polynomial time \([12]\).

Thus we have our main result:

**Theorem 2.** \( Q^p \) can be found in polynomial time.

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\(1\) A linear-time algorithm exists for the case \( k = 3 \) \([5,11]\).

\(2\) We suggest to solve the minimum-perimeter enclosing \( k \)-gon problem by reducing it to the shortest \( k \)-link path in simple polygon problem. The algorithms of \([12,14]\) for the latter are very non-trivial.

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**The “Flushness” Condition** We prove Lemma 1 with the standard method of “rotating calipers” \([16]\). Suppose \( Q^p \) is not flush with \( P \). Let \( C = \{p_{c_1}, \ldots, p_{c_k}\} \subset \{p_1, \ldots, p_n\} \) be the rocking points of the edges of \( Q^p \); \( C = bdQ^p \cap P \). Then the perimeter of \( Q^p \) is (refer to Fig. 2)

\[
p(Q^p) = \sum_{i=1}^{k} \frac{a_i}{\sin \alpha_i} (\sin \beta_i + \sin \gamma_i) = \sum_{i=1}^{k} \frac{a_i}{\sin \frac{\alpha_i}{2}} \cos \frac{\beta_i - \gamma_i}{2}
\]

Start rotating each edge of \( Q^p \) around its rocking point counterclockwise by an angle \( \theta \); call the polygon formed by such a rotation \( Q^p(\theta) \). Let \( \theta_{\text{min}}, \theta_{\text{max}} \) \((\theta_{\text{min}} < 0 < \theta_{\text{max}}) \) be the angles at which \( Q^p(\theta) \) becomes flush with \( P \). For \( \theta \in [\theta_{\text{min}}, \theta_{\text{max}}] \), \( Q^p(\theta) \) is still a feasible \( k \)-gon, enclosing \( P \). The perimeter of \( Q^p(\theta) \) as a function of \( \theta \) is

\[
f(\theta) \equiv p(Q^p(\theta)) = \sum_{i=1}^{k} \frac{a_i}{\sin \frac{\alpha_i}{2}} \cos \frac{(\beta_i - \theta) - (\gamma_i + \theta)}{2}
\]

Since for each \( i \), \( \beta_i - \theta \) and \( \gamma_i + \theta \) are angles of a triangle, \(|(\beta_i - \theta) - (\gamma_i + \theta)| < \pi \). Thus, as \( \cos(\cdot) \) is a concave function on \((-\pi/2, \pi/2)\), each summand in \((1)\) is a concave function of \( \theta \) for \( \theta \in (\theta_{\text{min}}, \theta_{\text{max}}) \). Hence, \( f(\theta) \) is also concave on \((\theta_{\text{min}}, \theta_{\text{max}}) \) and attains its minimum at one of the ends of the interval, i.e., when \( Q^p(\theta) \) is flush with \( P \). Q.E.D.

**Minimum-perimeter envelope** DePano and Aggarwal \([10]\) and Mount and Silverman \([13]\) considered the problem of finding the minimum envelope — an enclosing convex \( k \)-gon with a specified sequence of angles. The algorithms in \([10,13]\) for finding the minimum-area envelope \( Q^A_k \) are based (unsurprisingly) on the flushness condition. Note that our “flushness” Lemma 1 actually shows that the minimum-perimeter envelope \( Q^A_k \) is also flush with \( P \) and thus, can be

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\(3\) DePano \([9]\) and Chang \([7]\) in their theses proved the lemma for \( k = 3 \); the proof in \([9]\) uses a complicated trigonometric argument, the proof in \([7]\) is based on the similar result about minimum-area enclosing \( k \)-gon \([10]\). Our proof is different from those in \([7,9]\).
found in polynomial time. By following the algorithm of Aggarwal et al. for finding $Q_A^P$, we prove

**Theorem 3.** $Q_A^P$ can be found in $O(nk \log k)$ time.

**Restricted enclosures** In the original statement of the problem, the vertices of the enclosure were allowed to be placed just anywhere in the plane. We propose a generalization, in which two nested polygons $P_{out}$ and $P_{in} \subseteq P_{out}$ are given, and a minimum convex $k$-gon restricted to lie in between $P_{in}$ and $P_{out}$ is sought. Of course, the difference between unrestricted and restricted enclosures is that the latter may have some vertices on the boundary of $P_{out}$; following Bajaj we say that such vertices are “bash” with $P_{out}$. As far as we know, this generalization has not been studied before. The problem may be of interest in a classification task where the idea is to build a low-complexity separator between the data points of two types. We make a first small step in solving this type of problems by giving polynomial-time algorithms for finding minimum-area and minimum-perimeter restricted envelopes $Q_A^P$ and $Q_A^P$. Our solution is based on the fact that the optimal restricted polygons are “either flush or bash”.

**Lemma 4.** $Q_A^P$ is either flush with $P_{in}$ or is bash with $P_{out}$.

**Proof.** Otherwise, as in the proof of Lemma 1 start rotating each edge of the envelope around its rocking vertex of $P_{in}$ — the perimeter of $Q_A^P$ is a unimodal function of the turn angle and, thus, $Q_A^P$ may be rotated in one of the directions, decreasing its perimeter (Fig. 3).

Next we show that there is only a polynomial number of possible locations for the bash points.

**Lemma 5.** Suppose that a bash vertex $q_i$ of $Q_A^P$ and the edge $e$ of $P_{out}$ that $q_i$ lies on are given. Suppose that the edges $q_{i-1}q_i$ and $q_{i}q_{i+1}$ of $Q_A^P$ rock on the vertices $p_j$ and $p_l$ of $P_{in}$. Let $C$ be the circle through $p_j, p_l$ such that the segment $p_jp_l$ is seen at the angle $\alpha_i$ from the points on $C$; let $a_1, a_2$ be the points of intersection (if any) of $C$ with $e$. Then either $q_i = a_1$ or $q_i = a_2$ (Fig. 3).

**Theorem 6.** $Q_A^P$ may be found in $O(n_{in}^3n_{out})$ time, where $n_{in}$ and $n_{out}$ are the complexities of $P_{in}$ and $P_{out}$.

**Proof.** If $Q_A^P$ is flush with $P_{in}$, find $Q_A^P$ as in Theorem 2. Otherwise, for each triple $(p_j, p_l, q_i)$ (Fig. 3), $Q_A^P$ may be found by wrapping the envelope around $P_{in}$.

Similarly to Lemma 1 Mount and Silverman showed in that the area of the envelope as a function of the turn angle is unimodal. Thus, the above algorithm also works for finding $Q_A^P$.

**References**


