Every Combinatorial Polyhedron Can Unfold with Overlap

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Abstract

Ghomi proved that every convex polyhedron could be stretched via an affine transformation so that it has an edge-unfolding to a net [Gho14]. A *net* is a simple planar polygon; in particular, it does not self-overlap. One can view his result as establishing that every combinatorial polyhedron $\mathcal{P}$ has a metric realization P that allows unfolding to a net.

Joseph Malkevitch asked if the reverse holds (in some sense of "reverse"): Is there a combinatorial polyhedron $\mathcal{P}$ such that, for every metric realization P in $\mathbb{R}^3$, and for every spanning cut-tree T of the 1-skeleton, P cut by T unfolds to a net? In this note we prove the answer is NO: every combinatorial polyhedron has a realization and a cut-tree that edge-unfolds the polyhedron with overlap.

1 Introduction

Joseph Malkevitch asked[1] whether there is a combinatorial type $\mathcal{P}$ of a convex polyhedron P in $\mathbb{R}^3$ whose every edge-unfolding results in a net. One could imagine, to use his example, that every realization of a combinatorial cube unfolds without overlap for each of its 384 spanning cut-trees [Tuf11].[2] The purpose of this note is to prove this is, alas, not true: every combinatorial type can be realized and edge-unfolded to overlap: Theorem 2 (Section 5). For an overlapping unfolding of a combinatorial cube, see ahead to Figure 12.

An implication of Theorem 2, together with [Gho14], is that a resolution of Dürer's Problem [O'R13] must focus on the geometry rather than the combinatorial structure of convex polyhedra.

2 Proof Outline

We describe the overall proof plan in the form of a multi-step algorithm. We will illustrate the steps with an icosahedron before providing details.

Algorithm. Realizing G to unfold with overlap.
Input: A 3-connected planar graph G.
Output: Polyhedron P realizing G and a cut-tree T that unfolds P with overlap.

(1) Select outer face B as base.
(2) Embed B as a convex polygon in the plane.
(3) Apply Tutte's theorem to calculate an equilibrium stress for G.
(4) Apply Maxwell-Cremona lifting stressed G to P.
(5) Identify special triangle $\triangle$.
(6) Compress P vertically to reduce curvatures (if necessary).
(7) Stretch P horizontally to sharpen the apex of $\triangle$ (if necessary).
(8) Form cut-tree T, including '`Z`' around $\triangle$.
(9) Unfold $P \setminus T \to$ Overlap.

We are given a 3-connected planar graph G, which constitutes the combinatorial type of a convex polyhedron. By Steinitz's theorem, we know G is the 1-skeleton of a convex polyhedron. Initially assume G is triangulated; this assumption will be removed in Section 3.1.

(1) Select outer face B as base. Initially, any face suffices. Later we will coordinate the choice of B with the choice of the special triangle $\triangle$.

(2) Embed B as a convex polygon in the plane. Select coordinates for the vertices of B, which then pin B to the plane. B must be convex, but otherwise its shape is arbitrary.

(3) Apply Tutte's theorem [Tut63] to calculate an equilibrium stress—positive weights on each edge of G—that, when interpreted as forces, induce an equilibrium (sum to zero) at every vertex. This provides explicit coordinates for all vertices interior to B. The result is a Schlegel diagram, with all interior faces convex regions. Figure 1 illustrates this for the icosahedron.[3]

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[1]Personal communication, Dec. 2022.

[2]Burnside's Lemma can show that these 384 trees lead to 11 incongruent non-overlapping unfoldings of the cube [GSV19].

[3]Here the drawing is approximate, because I did not explicitly calculate the equilibrium stresses.

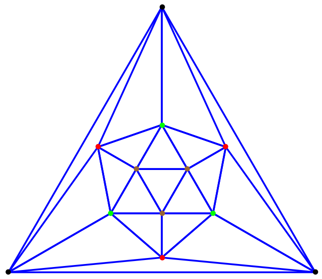

Figure 1: Icosahedron Schlegel diagram.

(4) Apply Maxwell-Cremona lifting to P. The Maxwell-Cremona theorem says that any straight-line planar drawing with an equilibrium stress has a polyhedral lifting via a "reciprocal diagram." The details are not needed here;[4] we only need the resulting lifted polyhedron. An example from [Sch08] shows the vertical lifting of a Schlegel diagram of the dodecahedron: Figure 2. A lifting of the vertices of the icosahedron in Figure 1 is shown in Figure 3.[5]

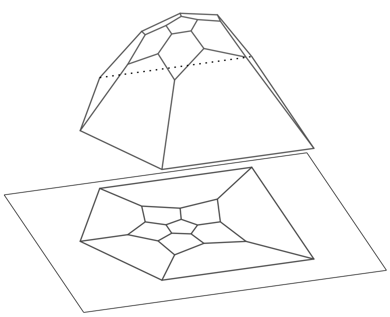

Figure 2: Maxwell-Cremona lifting to a dodecahedral diagram. [Sch08], by permission of author.

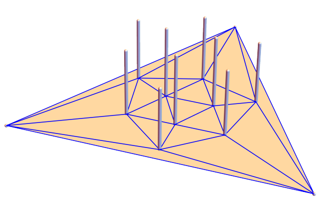

Figure 3: Vertical lifting the vertices of the icosahedron Schlegel diagram in Figure 1.

(5) Identify special triangle $\triangle$. This special triangle must satisfy several conditions, which we detail later (Section 3). For now, we select $\triangle = a_1a_2a_3 = 6, 8, 5$ in Figure 4.

[4]A good resource on this topic is [RG06].

[5]This is again an approximation as I did not calculate the reciprocal diagram.

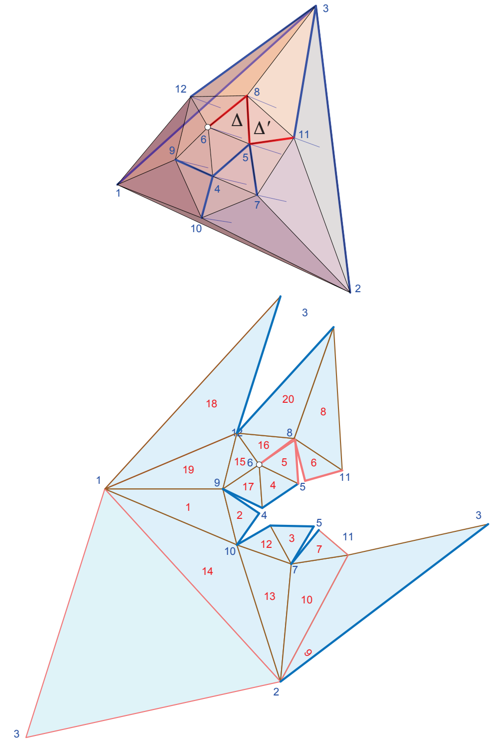

Figure 4: Red: face numbers; blue: vertex indices. $\triangle = 5$, $\triangle' = 6$. Z-portion of spanning tree T red; remainder blue.

(6) Compress P vertically (if necessary) to reduce curvatures. Not needed in icosahedron example.

(7) Stretch P horizontally (if necessary) to sharpen apex of $\triangle$. Not needed in icosahedron example.

(8) Form cut-tree T, including a 'Z'-path around $\triangle$. We think of a_1 as the root of the spanning tree, which includes the Z-shaped (red) path $a_1a_2a_3a_4$ around $\triangle$ and the adjacent triangle $\triangle'$ sharing edge a_2a_3. In Figure 4, the Z vertex indices are $6, 8, 5, 11$. The remainder of T is completed arbitrarily.

(9) Unfold $P \setminus T$. Finally, the conditions on $\triangle$ ensure that cutting T unfolds P with overlap along the a_2a_3 edge. See Figure 5.

3 Conditions on $\triangle$

We continue to focus on triangulated polyhedra. In order to guarantee overlap, the special triangle $\triangle = a_1a_2a_3$ should satisfy several conditions:

1. The angle at a_2 in $\triangle$ must be $\leq \pi/3 = 60°$, and the edge a_2a_3 at least as long as a_1a_2.

2. The spanning cut-tree T must contain the Z as previously explained. In addition, no other edge of T

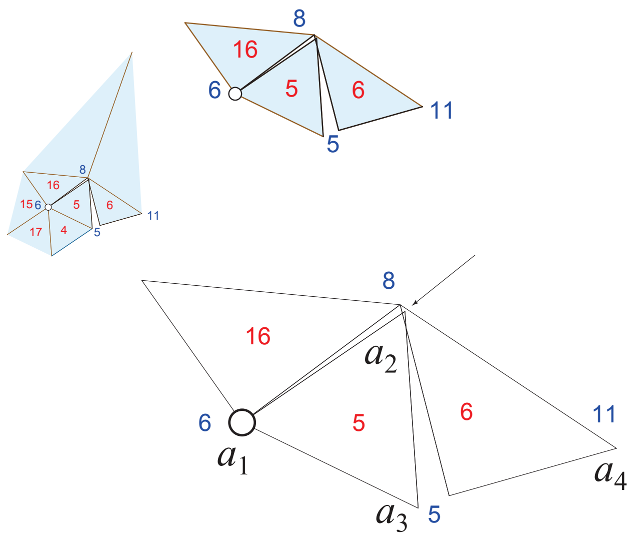

Figure 5: Close-up views of overlap.

is incident to either a_1 or a_2. In particular, edge a_1a_3 is not cut, so the triangle $\triangle$ rotates as a unit about a_1.

3. The curvatures at a_1 and a_2 must be small. (The *curvature* or "angle gap" at a vertex is 2π minus the sum of the incident face angles.) We show below that $< 20°$ suffices.

4. $\triangle$ should be disjoint from the base B: $\triangle$ and B share no vertices.

This 4th condition might be impossible to satisfy, in which case an additional argument is needed (Section 4). For now we concentrate on the first three conditions.

$\triangle$ is chosen to be the triangle disjoint from B with the smallest angle α. Clearly $\alpha \leq \pi/3 = 60°$. Let $\triangle = a_1a_2a_3$ with a_2 the smallest angle. Chose the labels so that $|a_1a_2| \leq |a_2a_3|$. It will be easy to see that $\triangle$ an equilateral triangle is the "worst case" in that smaller α lead to deeper overlap, and $|a_1a_2| = |a_2a_3|$ suffices for overlap. So we will assume $\triangle$ is an equilateral triangle.

Next, we address the requirement for small curvatures, when the second condition is satisfied: no other edge of T is incident to either a_1 or a_2. Let ω be the curvature at both a_1 and a_2. Then an elementary calculation shows that $\omega = \frac{1}{9}\pi = 20°$ would just barely avoid overlap: see Figure 6.

One can view the flattening of a_1 and a_2 when cut as first turning the edge a_2a_3 by ω about a_2, and then rotating the rigid path $a_1a_2a_3'$ about a_1 by ω. For any ω strictly less than $20°$, overlap occurs along the a_2a_3 edge: Figure 6(b). The basic reason this "works" to create overlap is that the cut-path around $\triangle$ is not *radially monotone*, a concept introduced in [O'R16] and used in [O'R18] and [Rad21] to avoid overlap.

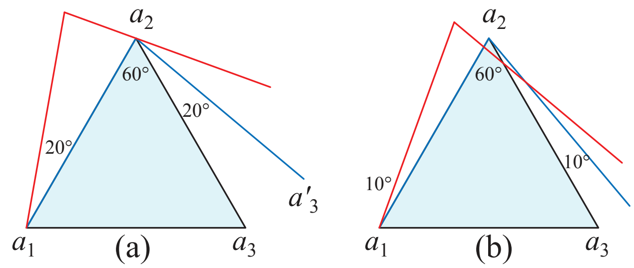

Figure 6: (a) $\omega = 20°$ avoids overlap. (b) $\omega = 10°$ overlaps.

In the unfolded icosahedron in Figure 4, the angle at a_2 is 59°, and the curvatures ω_1, ω_2 at a_1, a_2 are 2.4° and 8.1° respectively.

If the two curvatures are not less than 20°, then we scale P vertically, orthogonal to base B, step (6) of Algorithm 2. As illustrated in Figure 7, this flattens dihedral angles and reduces vertex curvatures (which reflect the spread of the normals [Hor84]) at all but the vertices of base B, which increase to compensate the Gsuss-Bonnet sum of 4π. Clearly we can reduce curvatures as much as desired.

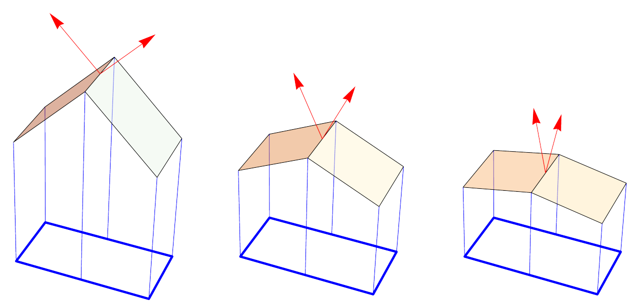

Figure 7: Dihedral angle δ flattens as z-heights scaled: $(1, \frac{1}{2}, \frac{1}{5}) \rightarrow (90°, 125°, 160°)$.

3.1 Non-Triangulated Polyhedra

If G and therefore P contains non-triangular faces, then we employ step (7) of Algorithm 1: Scale P horizontally, parallel to the xy plane containing B. For example, in the dodecahedron example (Figure 2), no face has an angle $\alpha \leq \pi/3$. The following lemma shows we can sharpen any selected face angle.

Lemma 1 *Any face angle $\angle a_1a_2a_3$ can be reduced via an affine stretching transformation to be arbitrarily small.*

Proof. Adjust the coordinate system so that a_1a_3 lies in the yz-plane containing the origin, with a_2 in the

x-positive halfspace, wlog at $a_2 = (1, a_{2y}, a_{2z})$. See Figure 8. The Tutte-embedding guarantees that $\triangle a_1a_2a_3$ is not degenerate—the three vertices are not collinear, and Maxwell-Corona lifting guarantees the triangle is not vertical because each vertex of the Schlegel diagram lies in the relative interior of its neighbors [RG06, p.126,136]. Now stretch all vertices by $s > 1$ in their x-coordinate. This leaves a_1 and a_3 fixed, while a_2 stretches horizontally to $a'_2 = (s, a_{2y}, a_{2z})$. Eventually with large s the angle $\angle a_1a'_2a_3$ decreases monotonically to zero, while maintaining $|a_1a'_2| \leq |a'_2a_3|$. □

So we can identify a $\triangle$ within any face, stretch its angle below 60°, and proceed just as in a triangulated polyhedron: Because a_1a_3 is not cut, having $\triangle$ joined to a triangle below (4 in Figure 4) is no different than having $\triangle$ part of a face.

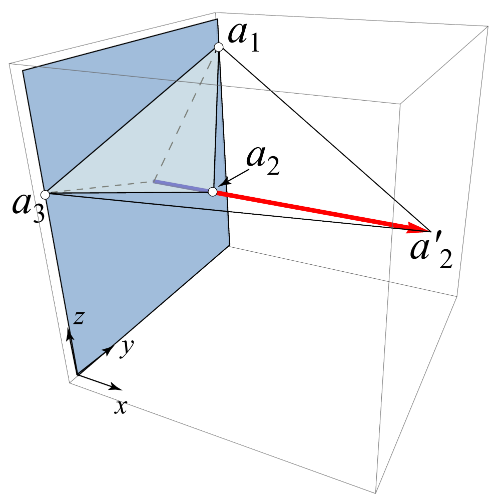

Figure 8: Stretching $\angle a_1a_2a_3 = 108°$ to $\angle a_1a'_2a_3 = 53°$.

4 No Pair of Disjoint Faces

Finally we focus on the 4th condition that $\triangle$ should be disjoint from the base B. If G contains any two disjoint faces, triangles or k-gon faces with $k > 3$, we select one as B and the other to yield $\triangle$. So what remains is those G with no pair of disjoint faces.

For example, a pyramid—a base convex polygon plus one vertex a (the apex) above the base—has no pair of disjoint faces. However, note that a pyramid has pairs of faces that share one vertex but not two vertices. It turns out that this suffices to achieve the same structure of overlap. Figure 9 illustrates why. Here B is a triangle $b_1b_2a_3$ and we select $\triangle = a_1a_2a_3$. The small-curvature requirement holds just for a_1, a_2—the start of the Z—the curvature at a_3 could be large (117° in this example) but does not play a role, as the unfolding illustrates. Therefore, if G has no pair of disjoint faces, but does have a pair of faces that share a single vertex, we proceed just in Algorithm 1, suitably modified.

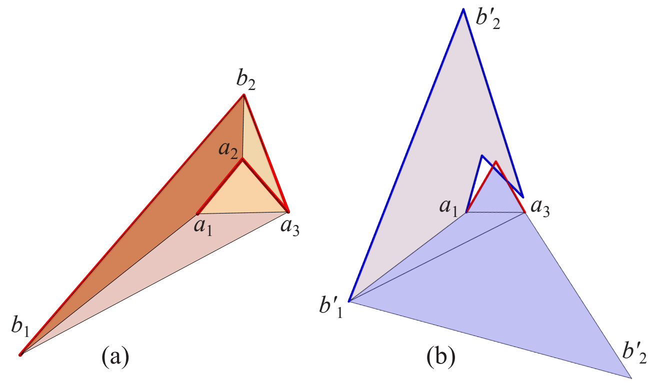

Figure 9: (a) B and $\triangle$ share a_3. $\mathtt{Z} = a_1a_2a_3b_2$. (b) Unfolding with overlap.

For the pyramid example, two triangles sharing just the apex would serve as $\triangle$ and base B. Consider the square pyramid in Figure 10(a), with B and $\triangle$ marked. Mapping $\triangle = 145$ to $\triangle = a_1a_2a_3$ at the shared pyramid apex, (b) of the figure shows that this is equivalent to Figure 9(a). A hexagonal pyramid is illustrated in the Appendix.

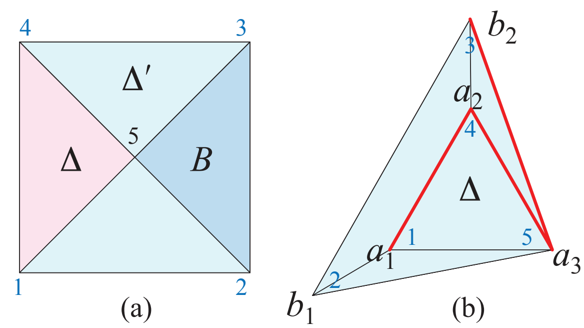

Figure 10: (a) Square pyramid Schlegel diagram, apex 5, square base 1234. (b) Relablled to match Figure 9(a).

This leaves the case where there are no two disjoint faces, nor two faces that share just a single vertex: every pair of faces shares two or more vertices. If two faces share non-adjacent vertices, they cannot both be convex. So in fact the condition is that each two faces share an edge. Then, it is not difficult to see that G can only be a tetrahedron, as the following argument shows.

Start with Euler's formula, $V - E + F = 2$. Each vertex v must be incident to exactly three faces, because, if v has degree ≥ 4, then each non-adjacent pair of faces incident to v cannot share an edge. So $3V = 2E$. Substituting into Euler's formula yields $F = 2 + E/3$.

Because each pair of faces share an edge, $F(F - 1)$

double counts edges:[6] $2E = F(F-1)$. Substituting,

$$\begin{aligned} F &= 2 + E/3 \\ E &= F(F-1)/2 \\ F &= 2 + F(F-1)/6 \\ F^2 - 7F + 12 &= 0 \end{aligned}$$

The two solutions of this quadratic equation are $F = 3$, which cannot form a closed polyhedron, and $F = 4$. The tetrahedron is the only polyhedron with four faces, and indeed $F = 4$ implies $V = 4$ and $E = 6$.

So the only case remaining is a tetrahedron. But it is well known that the thin, nearly flat tetrahedron unfolds with overlap: Figure 11. And since there is only one tetrahedron combinatorial type, this completes the inventory.

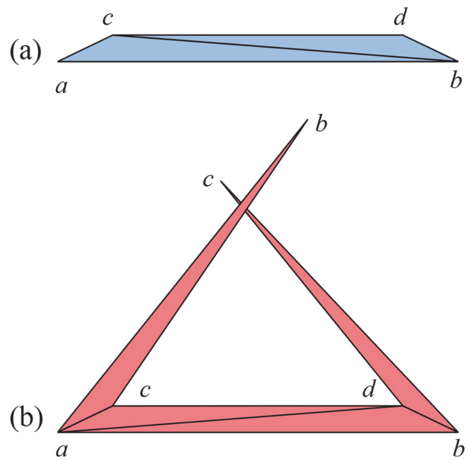

Figure 11: Figure 28.2 [detail], p.314 in [DO07]: tetrahedron overlap. Blue: exterior. Red: interior. Cut tree $T = abcd$. (T is a combinatorial 'Z'.)

5 Theorem

We have proved this theorem:

Theorem 2 *Any 3-connected planar graph G can be realized as a convex polyhedron P in $\mathbb{R}^3$ that has a spanning cut-tree T such that the edge-unfolding of $P \setminus T$ overlaps in the plane.*

So together with Ghomi's result,[7] any combinatorial polyhedron type can be realized to unfold and avoid overlap, or realized to unfold with overlap.

Returning to Malkevitch's example of a combinatorial cube, consider Figure 12. Starting from the standard Schlegel diagram for a cube (one square inside another (B), trapezoid faces between the squares), horizontal stretching (step (7) of Algorithm 1) is applied to squeeze the top and bottom squares to 1×2 and 2×4 diamonds, so that the angle at a_2 becomes small, in this case $2\arctan(1/2) \approx 53°$. The lifting leaves the curvatures at a_1, a_2 to be small enough, $6.0°, 6.5°$, so the vertical scaling step (6) of Algorithm 1 is not needed.

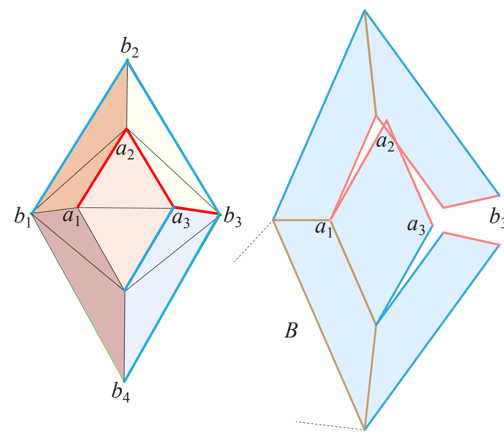

Figure 12: Unfolding of a combinatorial cube. Diagonals in the left figure are an artifact of the software; all lateral faces are planar congruent trapezoids. Base B attached left of b_1b_4 not shown. Vertex coordinates:

$$(-1, 0, 0.5), (1, 0, 0.5), (0, -2, 0.5), (0, 2, 0.5),$$

$$(-2, 0, 0), (2, 0, 0), (0, -4, 0), (0, 4, 0)$$

6 Open Problem

Is there a combinatorial type $\mathcal{P}$ of a Hamiltonian polyhedron (i.e., one with a Hamiltonian path), such that, for every metric realization $P \subset \mathbb{R}^3$, and every Hamiltonian path T, $P \setminus T$ unfolds to a net?

This restricts Malkevitch's question to combinatorial Hamiltonian polyhedra $\mathcal{P}$, and restricts T to a Hamiltonian path, producing a *zipper unfolding* [DDL+10]. Note: Some convex polyhedra are not Hamiltonian, e.g., the rhombic dodecahedron.

To rephrase the question: Is there a combinatorial Hamiltonian polyhedron whose every metric realization and zipper unfolding avoids overlap? Or is there instead an analog of Theorem 2 showing that even under these restrictions, there is always a realization and a zipper unfolding that overlaps?

Acknowledgements. I benefitted from discussions with Richard Mabry and Joseph Malkevitch, and three referees. In particular, one referee suggested Lemma 1 to repair an oversight.

[6] Similar logic is used to form Szilassi's polyhedral torus.
[7] See [SZ18] for a different proof of [Gho14].

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A Hexagonal Prism

Figure 13 shows a hexagonal prism, following the model of the square prism in Figure 10: no pair of faces are disjoint, but $\triangle$ and B marked share just one vertex. Figure 14 shows its overlapping unfolding.

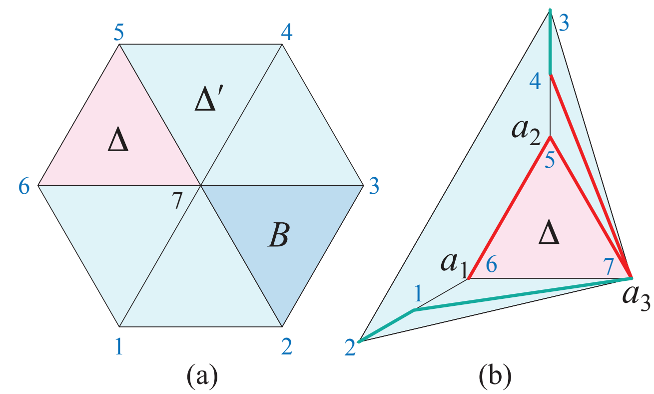

Figure 13: (a) Schlegel diagram of a hexagonal prism. (b) Overhead view of combinatorial rearrangement. The cut tree T is marked with red and blue paths.

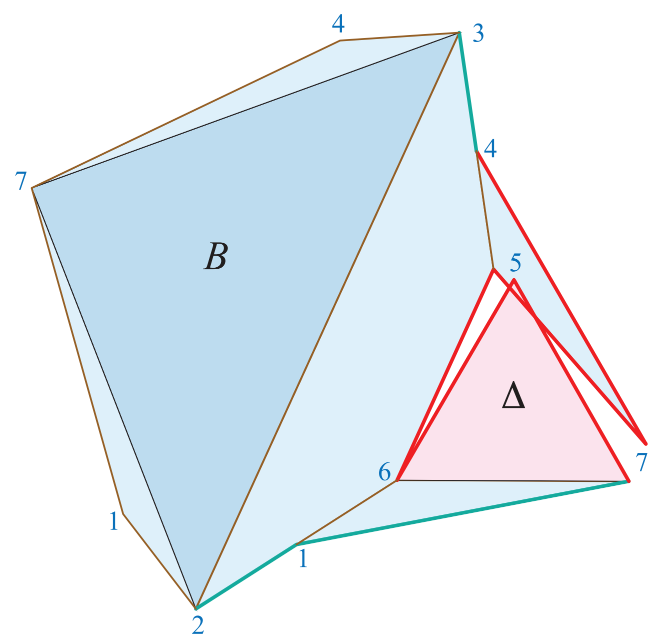

Figure 14: Unfolding of Figure 13(b). Curvature at v_6 and v_5 is 5.7°. Vertices 1 and 4 are collinear with 26 and 35 respectively. Hexagon: 123456.