Computational Geometry Column 50

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Abstract

Two long-open problems have been solved: (1) every sufficiently large planar point set in general position contains the vertices of an empty hexagon; (2) every finite collection of polygons of equal area have a common hinged dissection.

1 The Empty Hexagon Theorem

Geometric Ramsey theory results generally take the form of a claim that if some geometric structure is large or complex enough, then there must exist a nicely ordered substructure. One of the most beautiful theorems of this type was obtained by Erdös and Szekerez in 1935: for every k, there is an $N(k)$ such that every planar set of at least $N(k)$ points in general position contains the vertices of a convex k-gon [ES35]. $N(k)$ is known to be exponential in k, although its exact value remains a conjecture. About 40 years later, Erdös asked [Erd78] to find the smallest $H(k)$, if it exists, such that any planar set of at least $H(k)$ points in general position contains the vertices of an empty convex k-gon, one whose interior contains no points of the set. Empty triangles are unavoidable, and it is easily established that $H(4) = 5$. Harborth proved [Har79] that $H(5) = 10$, and Horton showed [Hor83] that $H(k)$ does not exist for all $k \geq 7$: there are arbitrarily large sets without empty heptagons. The last case, $H(6)$, has remained open since. Meanwhile the lower bound crept up via computer searches that found as many as 29 points without an empty hexagon [Ove03]. Now the question has been settled by two independent and nearly simultaneous proofs, by Nicolás [Nic07] and by Gerken [Ger08], that $H(6)$ exists. The best current bound is $H(6) \leq 1717$: any set with more points must contain an empty hexagon; see Fig. 1.

Both proofs start from the vertices of a minimal convex n-gon C_n guaranteed by the Erdös-Szekeres theorem: Nicolás uses a 25-gon and Gerken a 9-gon. Both then consider nested convex hull layers inside C_n, Nicolás penetrating four layers down and Gerken three layers (plus interior, which may be empty). Both proofs involve considerable case analysis. The Nicolás proof is shorter but obtains the weaker bound $H(6) \leq N(25)$; Gerken's proof methodically considers 57 cases but obtains $H(6) \leq N(9)$. The current best bound on $N(9)$ yields $H(6) \leq 1717$. Improvements on the proofs may be expected [Val07][Kos07].

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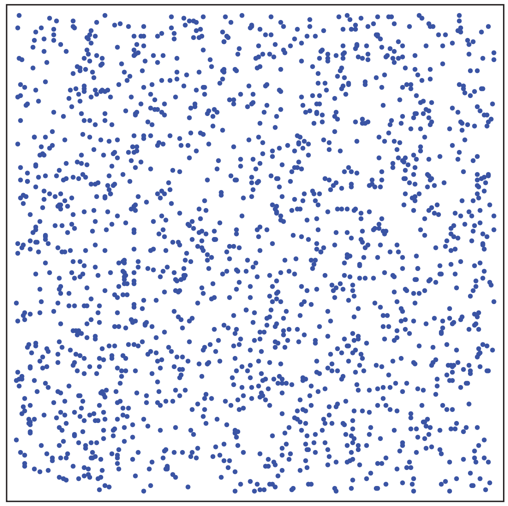

Figure 1: Among these 1717 points, there must be a convex 9-gon that contains an empty hexagon.

2 Hinged Dissections Exist

In *Computational Geometry Column 44* [O'R02], I reported on progress on the open problem of determining whether hinged dissections exist: Does every pair of equal-area polygons have a hinged dissection? More generally, does every collection of polygons of equal area have a common hinged dissection? A *dissection* partitions each collection into a finite number of pieces and reassembles via rigid motions to form the other collection. It has been known since the early 19th century that this is always possible for dissections [Fre97], but the same question for hinged dissections has remained unresolved. *Hinged dissections* hinge the pieces together at vertices into a connected assembly, whose elegant motions have attracted the attention of puzzle enthusiasts for over a century. Now the question is settled completely in a paper by Abbott et al. [AAC+07]: any finite set of polygons of equal area have a common hinged dissection. The result goes beyond bare existence to establish two attractive properties of the dissections. First, if one builds a physical model of a hinged dissection (see [Fre02] for examples of such models), the movement is most natural if the pieces do not properly intersect during the hinging motions. The theorem establishes this non-intersection property. Second, the theorem achieves a pseudopolynomial number of pieces (for a constant number of polygons with vertices on a rational grid), which is best possible in the worst case.

The proof starts from a common dissection of the two collections of polygons, and hinges the pieces together into two (different) tree-like structures. A key move is reorganizing the structure of one tree, taking a step toward the structure of the other. The basic construction is illustrated in Fig. 2.

Avoiding self-intersections during the motion employs the "slender adornments" theory developed in [CDD+06], and achieving a pseudopolynomial number of pieces uses an idea in [Epp01].

Finally, Abbott et al. were able to extend their result to three dimensions for pairs of polyhedra that are equidecomposable (not all are, as Dehn showed), although here the motion is not guaranteed to avoid self-intersection.

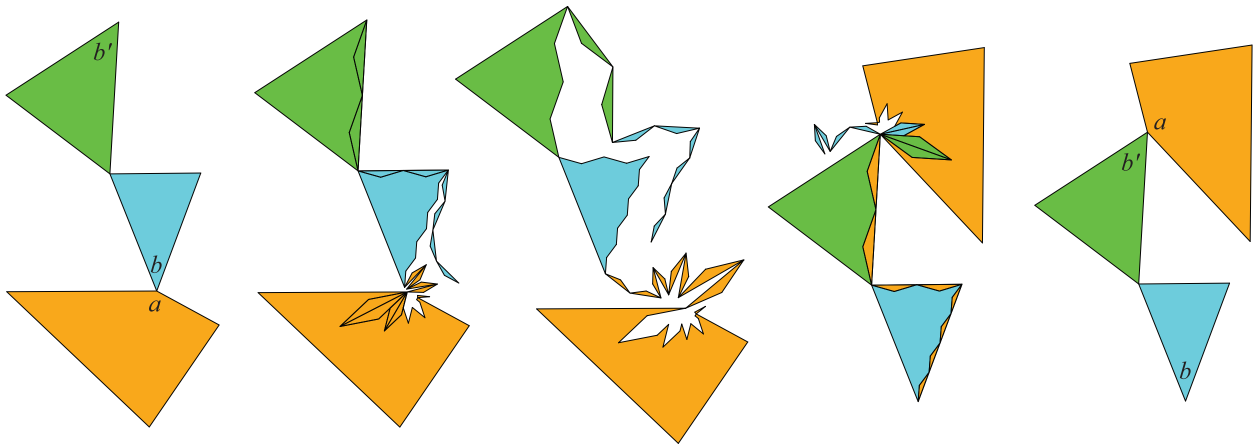

Figure 2: Moving a subtree rooted at a, originally attached to b (left), so that it is instead attached to b' (right). [Based on Fig. 5 of [AAC+07], with permission.]

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