Computational Geometry Column 49

Joseph O'Rourke*

Abstract

The new algorithm of Bobenko and Izmestiev for reconstructing the unique polyhedron determined by given gluings of polygons is described.

One form of Cauchy's rigidity theorem states that the combinatorial structure of a triangulated convex polyhedron together with all its edge lengths determines a unique convex polyhedron P: the 3D vertex coordinates are uniquely determined (up to rigid motions) by this information. However, it has long been an unsolved problem to algorithmically reconstruct the geometric shape.

Sabitov found an exponential algorithm to solve this problem based on the "volume polynomial" [Sab96]. A recent extension of Sabitov's work [FP05] establishes that the unknown internal diagonal lengths between each pair of vertices are roots of a polynomial of degree at most 4^m, where m is the number of edges of the polyhedron P. Knowing these internal diagonals permits reconstruction. Exponential lower bounds on the polynomial degree left practical reconstruction unresolved.

One can view the information input to Cauchy's result as a gluing-together of a collection of polygons to form a topological sphere. In Cauchy's theorem, the polygons are in fact the faces of the polyhedron. Alexandrov [Ale05] proved a significant strengthening: any gluing-together of a collection of polygons to form a topological sphere leads to a unique convex polyhedron,[1] as long as no more than 2π of angle is glued at any one point. The polygons now have no relation to the faces. In fact, his theorem applies even to just one polygon whose perimeter is glued to, or identified with itself. Alexandrov's proof is, alas, nonconstructive, and did not suggest an algorithm.

Now Alexander Bobenko and Ivan Izmestiev have found a constructive proof of Alexandrov's theorem, which leads to an effective numerical algorithm to reconstruct the 3D structure of the unique polyhedron guaranteed by the theorem, and therefore of Cauchy's theorem as well [BI06]. They and Stefan Sechelmann have implemented the algorithm and have made it available through a Java applet.[2]

The historical roots of their algorithm go back to a remark made by Blaschke and Herglotz in 1937 concerning a related problem posed by Weyl. Alexandrov speculated on the form a

*Dept. of Computer Science, Smith College, Northampton, MA 01063, USA. orourke@cs.smith.edu.

[1] Where a flat, doubly-covered polygon is considered a convex polyhedron.

[2] `http://www.math.tu-berlin.de/geometrie/ps/software.html`

constructive proof might take in 1950 [Ale05, p. 320–1], and his student Volkov carried out a portion of the project in 1955, now included as Appendices in [Ale05]. The new result can be viewed as a completion of this project.

The gluing-together of polygons mentioned above defines a topologically spherical surface S with a *convex polyhedral metric*: a metric d so that every point $x \in S$ is either flat, i.e., with a neighborhood isometric to a subset of $\mathbb{R}^2$, or a vertex, i.e., with a neighborhood isometric to an open subset of a cone with angle $< 2\pi$ at the apex x. The polygon gluing defines (S, d). The challenge is to construct the unique polyhedron P (e.g., the 3D coordinates of its vertices) guaranteed by Alexandrov's theorem (or Cauchy's theorem if the polygons are the faces of P).

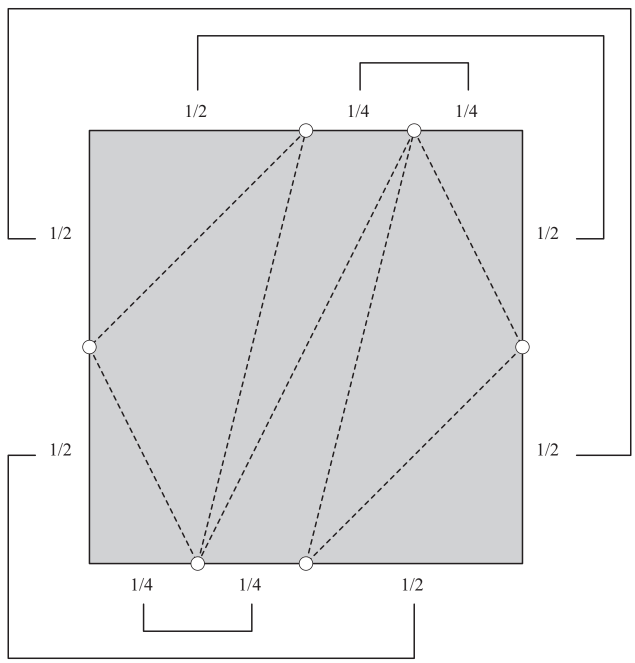

Figure 1: Folding of a unit square produces a polyhedron with 6 vertices (marked) and 8 triangle faces. Lengths of edges on the perimeter are as indicated, and edges connected by an exterior path are identified, i.e., glued to one another.

As an example, consider the gluing of the perimeter of a unit square to itself indicated in Fig. 1. The metric defined by this gluing has 6 vertices, those points at which the total surrounding angle is $< 2\pi$. Alexandrov's theorem only requires this gluing as "input." The dashed lines indicate the creases of the folding into 3D, which I know from applying ad hoc techniques for octahedron reconstruction [ADO03][DO07, Ch. 25]. Even knowing these, which now provide the "input" for Cauchy's theorem, it is far from obvious what is the exact shape of the convex polyhedron this folding produces. The difficulty is that the resulting octahedron has an internal diagonal of unknown length. (The latest version of Sabitov's algorithm already requires finding roots of a polynomial of degree 4^m=4^{12}=$16,777,216$ to find this length.)

The basic idea of the new algorithm is to extend the metric on the surface S to the interior (the Blaschke-Herglotz suggestion), producing a metric structure on the ball. This is accomplished by starting with a "solid" object that cannot be embedded in $\mathbb{R}^3$, and deforming it according to a differential equation until it becomes, in the limit, $P \subset \mathbb{R}^3$. The equation is a function of t running from $t = 1$ to $t = 0$.

The algorithm starts with a *geodesic triangulation* $T(1)$ of S: a partition of S into triangles whose edges are geodesic paths on S, and whose vertices are the cone points of S. More on how $T(1)$ is selected in a moment. At any time t, a *generalized polytope* is defined by an apex a and radii of length r_i from a to each vertex of $T(t)$. If $\triangle ijk$ is one face of $T(t)$, then r_i, r_j, and r_k form a pyramid over $\triangle ijk$. All these pyramids share the same apex a. Let κ_i, the *curvature* at r_i, be 2π minus the sum of the dihedral angles of the pyramids sharing r_i. At the $t = 0$ endpoint P, we must have $\kappa_i = 0$ for all i, for then the pyramids fill the interior of P, forming a solid topological ball, and so completely surround each radial spoke. Generalized polytopes permit $\kappa_i \neq 0$, which implies they cannot be embedded in $\mathbb{R}^3$.

The starting triangulation $T(1)$ is a crucial ingredient of their algorithm, described in another paper [BS05]. It is the Delaunay triangulation defined via intrinsic geodesic distance on S. One way to compute the Euclidean Delaunay triangulation of a set of points in the plane is to start with an arbitrary triangulation, and "flip" any edge e that is not *locally Delaunay*: where a vertex of one triangle sharing e falls inside the circumcircle of the other triangle sharing e. It is a remarkable fact that repeatedly flipping non-locally Delaunay edges converts any triangulation into the Delaunay triangulation (in at most $O(n^2)$ flips) [Ede01]. It is even more remarkable that the generalization to geodesic triangulations holds: the flipping algorithm constructs the geodesic Delaunay triangulation in a finite number of steps. Two aspects make this a nontrivial extension of the planar result. First, an empty disk bounded by a circumcircle might wrap around S and self-overlap. Second, S in general has an infinite number of geodesic triangulations. Even a cube has an infinite number of such triangulations, as can be seen as follows. Partition the top and bottom cube faces by diagonals, and view the vertical faces as determining a cylinder. Now one can spiral noncrossing geodesics around this cylinder connecting the bottom and top vertices forming an infinite variety of "barber pole" triangulations. This complication makes it impossible to bound the number of flips, although the number is finite.

A *generalized convex polytope* requires, in addition to the r_i defining pyramids, each edge ij of the triangulation to have a dihedral angle θ_{ij} (determined by the pyramids) of $\leq \pi$, i.e., compatible with convexity. Bobenko and Izmestiev prove that, for the Delaunay geodesic triangulation of S, there is a sufficiently large R such that setting all $r_i = R$, i.e., $\mathbf{r}(1) = (R, R, \ldots, R)$, defines a generalized convex polytope $P(1)$, which can serve as the starting point. They also prove that a set of r_i determine a unique generalized convex polytope, important for intermediate steps.

Now the stage is set for deformation of $P(1)$ to $P(0)$. The idea is to drive each κ_i to zero via $\kappa_i(t) = t \cdot \kappa_i(1)$ for t going from 1 to 0. As the κ_i change, the r_i deform according to a differential equation

$$\frac{d\mathbf{r}}{dt} = M(t) \cdot \kappa(1) \ , \quad M(t) = \left[\frac{\partial \kappa_i}{\partial r_j} \right]^{-1} \ .$$

Here $\mathbf{r}$ and $\kappa(1)$ are both n-vectors, and $M(t)$ is the $n \times n$ inverse matrix of the Jacobian computed for the polytope $P(t)$. Establishing the nondegeneracy of the Jacobian is a crucial aspect of the proof, complicated by the fact that the geodesic triangulation $T(t)$ on which the terms are based is changing via flips throughout the process. In the end, $\lim_{t \to 0} \mathbf{r}(t) = \mathbf{r}(0)$ exists and determines $P(0) = P$, the unique polyhedron realizing the given polyhedral metric (S, d).

Their software follows this recipe, computing $T(1)$, and then numerically solving the differential equation. Fig. 2 shows a screen snapshot of the software reconstructing the octahedron from the gluing defined in Fig. 1.

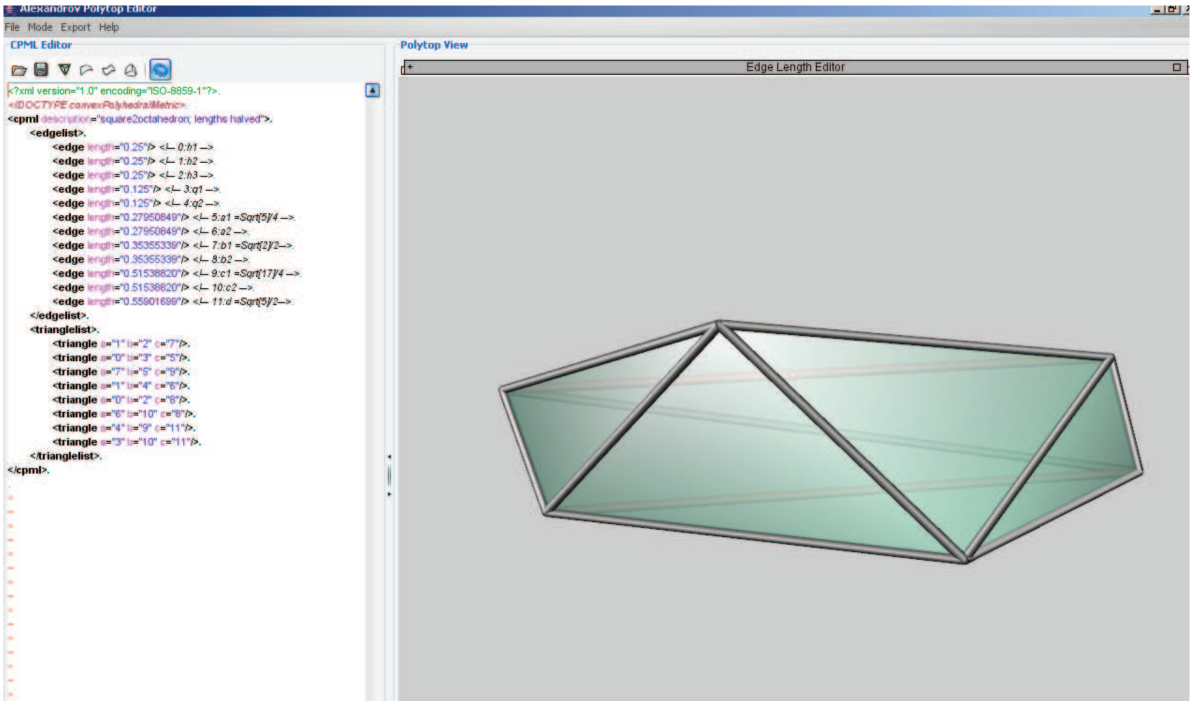

Figure 2: Screen shot of the software. To the left is the XML file describing the triangle gluings, to the right the reconstructed 6-vertex octahedron.

Because the starting point of this algorithm is determined by flipping to the Delaunay geodesic triangulation, and because the number of flips cannot be bounded, the algorithm is not polynomial-time in the number of vertices n. But it remains pragmatically effective, as the software demonstrates.

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